

Metric Geometry and Collapsibility

Karim Adiprasito ^{*}
 Inst. Math., FU Berlin
adiprasito@math.fu-berlin.de

Bruno Benedetti ^{**}
 Dept. Math., KTH Stockholm
brunoben@kth.se

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Abstract

Cheeger’s finiteness theorem bounds the number of diffeomorphism types of manifolds with bounded curvature, diameter and volume; the Hadamard–Cartan theorem, as popularized by Gromov, shows the contractibility of all non-positively curved simply connected metric length spaces. We establish a discrete version of Cheeger’s theorem (*“In terms of the number of facets, there are only exponentially many geometric triangulations of Riemannian manifolds with bounded geometry”*), and a discrete version of the Hadamard–Cartan theorem (*“Every complex that is CAT(0) with a metric for which all vertex stars are convex, is collapsible”*). The first theorem has applications to discrete quantum gravity; the second shows that Forman’s discrete Morse theory may be even sharper than classical Morse theory, in bounding the homology of a manifold. In fact, although Whitehead proved in 1939 that all PL collapsible manifolds are balls, we show that some collapsible manifolds are not balls.

Further central consequences of our work are:

- (1) Every flag connected complex in which all links are strongly connected, is Hirsch.
(This strengthens a result by Provan–Billera.)
- (2) Any linear subdivision of the d -simplex collapses simplicially, after $d - 2$ barycentric subdivisions. (This presents progress on an old question by Kirby and Lickorish.)
- (3) There are exponentially many geometric triangulations of S^d .
(This interpolates between the result that polytopal d -spheres are exponentially many, and the conjecture that all triangulations of S^d are exponentially many.)
- (4) If a vertex-transitive simplicial complex is CAT(0) with the equilateral flat metric, then it is a simplex. (This connects metric geometry with the evasiveness conjecture.)
- (5) The space of phylogenetic trees is collapsible.
(This connects discrete Morse theory to mathematical biology.)

Contents

1	Introduction	2
2	Preliminaries	6
2.1	CAT(k) spaces and (strongly) convex subsets	6
2.2	Spaces of curvature bounded below and Cheeger’s theorem	7
2.3	Simplicial complexes, geometric triangulations and paths	7
2.4	Rocket shellings, vertex decomposability and the Hirsch Conjecture	10
2.5	Acute and nonobtuse triangulations	11

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3	Main Results	12
3.1	Gradient Matchings and Star-Minimal Functions	12
3.2	Discrete Hadamard–Cartan Theorem	15
3.3	Discrete Cheeger Theorem	18
3.4	Complexes with convex geometric realizations	22
4	Applications	27
4.1	Vertex-transitive triangulations	27
4.2	Flag manifolds are Hirsch	28
4.3	A collapsible manifold that is not a ball	30

1 Introduction

Whitehead, Cheeger and Gromov extolled the interplay of metric geometry, topology and combinatorics. Whitehead’s approach was motivated by Poincaré’s conjecture that homotopy recognizes spheres among all closed manifolds. In contrast, Whitehead first realized that homotopy is insufficient to recognize balls among all manifolds with boundary. In fact, for each $d \geq 4$, there are contractible smooth d -manifolds whose boundary is not simply-connected, hence they are not homeomorphic to the d -ball [61, 64, 79].

In 1939, Whitehead proposed an algorithmic way to understand the notion of contractibility, called “*collapsibility*”. All collapsible complexes are contractible, but the converse is false [15]. The collapsibility property is of particular interest when applied to triangulations of manifolds. Even if not all contractible manifolds are balls, Whitehead showed that all *collapsible PL triangulations* of manifolds are homeomorphic to balls [80, Corollary 1, p. 293]. PL stands for Piecewise Linear and represents a technical combinatorial requirement, which in particular demands the star of every vertex to be a ball; all smooth manifolds admit PL triangulations [69]. The PL property is not implied by collapsibility: A first example E of a non-PL 5-ball was found by Edwards in the Seventies [33], and the cone over this E yields a collapsible non-PL triangulation of the 6-ball. However, so far it has always been unclear whether the PL assumption is really necessary for Whitehead’s result.

Problem 1. *Are all collapsible triangulations of manifolds homeomorphic to balls?*

Problem 1 can be seen as a discretization issue. If we are given a smooth manifold M , classical Morse theory provides upper bounds for the Betti numbers of M , via the Morse inequalities. In 1999, Forman introduced a discretization of Morse theory, based on Whitehead’s approach [37]. After choosing some triangulation of M , discrete Morse theory yields also upper bounds for the Betti numbers of M , via the discrete Morse inequalities. If we are free to pick any triangulation and any smooth structure we want, which theory gives the sharper bounds? In case M is contractible, this question boils down to Problem 1, because

- collapsible triangulations are the only ones admitting a sharp *discrete* Morse function, and
- balls are the only contractible manifolds admitting a sharp *smooth* Morse function (in the sense of cobordisms, cf. Sharko [73]).

There was some evidence in favour of a *positive* answer for Problem 1. Contractible shellable manifolds are indeed balls. The second author recently introduced a dual notion to collapsibility, called ‘endo-collapsibility’, and proved that also all contractible endo-collapsible manifolds are balls [11, Theorem 3.12]. However, endo-collapsibility and collapsibility are not equivalent properties, and shellability is slightly stronger than collapsibility.

On the other hand, if a *negative* solution for Problem 1 exists, the main difficulty in finding collapsible triangulations with exotic topologies is that we lack good criteria to understand when a complex collapses. The few available criteria, like

- (a) “all cones are collapsible”,
- (b) (*Chillingworth’s theorem* [25]) “all convex 3-complexes are collapsible”, and
- (c) (*Crowley’s criterion* [29]) “all 3-dimensional pseudomanifolds that are CAT(0) with the equilateral flat metric, are collapsible”,

are all useless for our purpose, because

- (a) the cone over a manifold M is a manifold if and only if M is either a sphere or a ball,
- (b) every convex simplicial complex is homeomorphic to a ball, and
- (c) all 3-manifolds are PL.

We will come back to Chillingworth’s result towards the end of this Introduction. Crowley’s criterion is a first “*combinatorial analog of Hadamard’s theorem*” [29, p. 36], and deserves some extra explanation. CAT(0) is a property of metric spaces, popularized by Gromov’s work [43]. Roughly speaking, CAT(0) spaces are metric spaces where all triangles look thinner than corresponding triangles in the Euclidean plane; the Hadamard–Cartan theorem guarantees that such spaces are always contractible. “CAT(0) with the equilateral flat metric” is instead a property of simplicial complexes. It means that if we give the complex a piecewise-Euclidean metric by assigning unit length to all its edges, then the complex becomes a CAT(0) metric space.

In the present paper, building on this connection between metric geometry and combinatorics, we reach the solution of Problem 1:

Main Theorem 1 (Theorem 4.12). *For each $d \geq 5$, some smooth d -manifolds different from balls admit collapsible triangulations.*

The bound $d \geq 5$ is the best possible: If $d \leq 4$, all triangulations of d -manifolds are PL, so those that are also collapsible are automatically homeomorphic to balls, by Whitehead’s work [80]. Main Theorem 1 shows that Forman’s discretization of Morse theory is sometimes sharper than the original theory, in bounding the homology groups of a manifold.

The key to reach Main Theorem 1 is a new geometric criterion for collapsibility, which works in all dimensions. It yields a powerful discrete analog of the Hadamard–Cartan theorem.

Main Theorem 2 (Theorem 3.8). *Every complex that is CAT(0) with a metric for which all vertex stars are convex, is collapsible.*

The *convexity of vertex stars* means that for each vertex v , with respect to the metric introduced, the shortest path between any two points of $|\text{star } v|$ lies entirely in $|\text{star } v|$. This condition cannot be removed: All triangulated 3-balls are CAT(0) with a suitable metric, but some 3-balls are not collapsible. However, this condition is automatically satisfied by any complex in which all simplices are acute or right-angled. In particular, any complex that is CAT(0) with the equilateral flat metric, has also convex vertex stars; so by Main Theorem 2 it is collapsible. Therefore, Main Theorem 2 extends Crowley’s criterion to all dimensions, to all complexes (not only pseudomanifolds), and to an infinite range of metrics different than the equilateral flat one. For example, the “space of phylogenetic trees” introduced by Billera, Holmes and Vogtmann [14] turns out to be collapsible (Corollary 3.17).

The proof of Main Theorem 2 is as follows. Crowley’s method to pick a vertex v and then to draw a Morse matching “towards v ”, as in [29], may fail. So we come up with a more delicate, metric approach: We look at the point m of the underlying space $|\text{star } \sigma|$ where the function “distance from v ” attains a local minimum. The convexity assumption guarantees that m is unique; however, it might not be a vertex. In case m is not a vertex and it does not belong to σ , we approximate m with a vertex y_m of $\text{star } \sigma$, and then we match the face σ with the smallest face containing both σ and y_m . If we apply this rule to all faces σ , starting from the lower-dimensional ones, we obtain the correct Morse matching; the verification is a bit long, but otherwise elementary, cf. Section 3.1.

The converse of Main Theorem 2 is false. Being CAT(0) with the equilateral flat metric is a stronger property than being collapsible; so stronger, that it forces any vertex-transitive complex with this property to be a simplex. This opens doors to the Evasiveness Conjecture of theoretical computer science, cf. Section 4.1.

Another important combinatorial conjecture, recently solved (in the negative) by Santos [71], is whether all polytope boundaries are Hirsch. Recall that a pure d -complex with n vertices is called *Hirsch* if the diameter of its dual graph is bounded above by $n - d - 1$. The work by Provan and Billera has revealed that the barycentric subdivision of any shellable sphere (a class which contains all polytope boundaries, but not all spheres) is Hirsch [67]. What about the barycentric subdivisions of the non-shellable spheres? Or of different manifolds, say? Here our metric approach is natural, since the Hirsch property is a combinatorial distance bound.

Main Theorem 3 (Cor. 4.7). *Every flag connected complex in which all links are strongly connected, is Hirsch. In particular, all barycentric subdivisions of homology manifolds are Hirsch.*

“Flag” means that the minimal non-faces are edges. Gromov showed that when endowed with a right-angled spherical metric, a simplicial complex is CAT(1) if and only if it is flag [43, Theorem 4.2.A]. The proof of Main Theorem 3 relies on Gromov’s result: Given two arbitrary facets, in order to establish their “combinatorial distance” in the sense of Hirsch, we approximate a geodesic connecting them with a combinatorial path in the dual graph.

The number of triangulations: a problem from quantum gravity

With these tools in our hand, we can turn to an important issue in discrete quantum gravity. In the so-called *Regge Calculus*, introduced in the Sixties, one approximates Riemannian structures on a smooth manifold M by considering all *triangulations* of M [68]. The metric is introduced on each triangulation a posteriori, by assigning to each edge a certain length. For example, Weingarten’s *dynamical triangulations* approach assigns to all edges the same length [4, 77]. This is exactly what we previously called “equilateral flat metric”.

These models gained vast popularity in the recent decades, due to their tremendous simplification power: For example, the *partition function* for quantum gravity, a path integral over all Riemannian metrics, becomes a sum over all possible triangulations with N facets [4]. Yet to make sure that this sum converges when N tends to infinity, one needs to provide an exponential bound for the number of triangulated d -manifolds with N facets. There lies the rub: Already for $d = 2$, there are more than exponentially many surfaces with N triangles (cf. Example 3.19). For $d = 2$ the problem can be bypassed by restricting the topology, because for fixed g there are only exponentially many triangulations of the genus- g surface, cf. [4] [74].

In dimension greater than two, however, the situation is unclear. Among others, Gromov [44, pp. 156–157] asked whether there are more than exponentially many triangulations of S^3 ; the problem is still open. Part of the difficulty is that when $d \geq 3$ many d -spheres cannot be realized as boundaries of $(d + 1)$ -polytopes [51], and cannot be shelled [46]. We do know that polytopal and shellable spheres are only exponentially many [13].

Keeping in mind the original, physical motivations of the problem, we want to study the question from a more geometric point of view.

Problem 2. *Are there metric conditions for d -manifolds that force exponential bounds for the number of triangulations of manifolds satisfying the conditions?*

This approach was attempted already in 1996 by a group of physicists [9], but their work is in many aspects (and in their own words) incomplete. We tackle Problem 2 by combining our Main Theorem 2 with Cheeger’s finiteness theorem, which states that there are only finitely many

diffeomorphism types of Riemannian manifolds with “bounded geometry” (namely, curvature and volume bounded below, and diameter bounded above). What we achieve is a discrete analog of Cheeger’s theorem, which (roughly speaking) shows that *triangulations* of manifolds with bounded geometry are very few, compared to all triangulations of all manifolds.

Main Theorem 4 (Theorem 3.27). *In terms of the number of facets, there are exponentially many triangulations of Riemannian manifolds with bounded geometry (and fixed dimension).*

Here is the proof idea: Via Cheeger’s bounds on the injectivity radius, we can chop any manifold of constant curvature into a constant number of convex pieces. These pieces have small diameter, and it is not hard to extend Main Theorem 2 from $\text{CAT}(0)$ spaces to $\text{CAT}(k)$ spaces of small diameter. So we conclude that all these pieces are collapsible; in particular, their barycentric subdivisions are endo-collapsible. As a result, we obtain a discrete Morse function on the whole manifold with a bounded number of critical $(d - 1)$ -cells. This implies the desired exponential bounds, via the results in [11, 13].

Finally, we study a fixed-metric variant to Problem 2. Inspired by Gromov’s question on S^3 , let us consider the unit sphere S^d with its standard intrinsic metric. A “geometric triangulation” is a tiling of S^d into regions that are convex simplices with respect to the given metric, and combinatorially form a simplicial complex.

Problem 3. *How many geometric triangulations of the standard S^d are there into N convex regions?*

We are interested in Problem 3 because all boundaries of simplicial $(d + 1)$ -polytopes can be seen as geometric triangulations of S^d . Except when $d \leq 2$, the converse is false: There are more geometric triangulations than polytope boundaries, cf. Example 2.7. We know that polytopal d -spheres are only exponentially many. Perhaps geometric triangulations are also exponentially many? Using Chillingworth’s theorem and the ideas of Main Theorem 4, it is not difficult to confirm this intuition for geometric triangulations of S^2 and of S^3 .

In dimension ≥ 4 , however, there is a difficulty: Chillingworth’s theorem is no longer available. In fact, whether all convex d -balls are collapsible represents a long-standing open problem, asked among others by Chillingworth, Goodrick, Lickorish and Kirby, cf. [25] [52, Problem 5.5]. The new result of Section 3.4 is the following: We show that Chillingworth’s result can indeed be generalized to higher dimensions, but in a slightly weaker form than the one generally expected.

Main Theorem 5 (Theorem 3.42). *If a d -complex admits a geometric realization as convex subset of \mathbb{R}^d , then its $(d - 2)$ -nd barycentric subdivision is collapsible.*

Indeed, when d is fixed, what is exponential in terms of $(d + 1)!^{d-2} N$ is also exponential in terms of N . So, instead of counting simplicial complexes of dimension d with N facets, we may as well count the $(d - 2)$ -nd barycentric subdivisions of these complexes, in terms of their number $(d + 1)!^{d-2} N$ of facets. In other words, we can replace each complex with its $(d - 2)$ -nd barycentric subdivision, as this affects the counting only by an exponential factor. This trick explains why Main Theorem 5 is all we need, in order to successfully tackle Problem 3.

Main Theorem 6 (Corollary 3.45). *There are exponentially many geometric triangulations of the standard S^d into N convex regions.*

Our methods use, of course, convex and metric geometry; whether *all* triangulations of S^d are exponentially many, remains an open problem. But even if the answer will turn out to be negative, Main Theorems 4 and 6 provide concrete mathematical support for the hope of discretizing quantum gravity in *all* dimensions.

2 Preliminaries

2.1 CAT(k) spaces and (strongly) convex subsets

All of the metric spaces we consider here are *compact, connected length (metric) spaces*. For a more detailed introduction, we refer the reader to the textbook by Burago–Burago–Ivanov [18].

Given two points a, b in a length space X , we denote by $|ab|$ the *distance* between a and b , which is also the infimum (and by compactness, the minimum) of the lengths of all paths from a and b . A *geodesic* from a to b is a curve γ from a to b which is locally a shortest path. (This means, each $t \in [0, 1]$ has a neighborhood $J \subset [0, 1]$ such that $\gamma|_J$ is the shortest path between its endpoints.) A *geodesic triangle* in X is given by three vertices a, b, c connected by shortest paths $[a, b]$, $[b, c]$ and $[a, c]$.

Let k be a real number. Depending on the sign of k , by *the k -plane* we mean either the Euclidean plane (if $k = 0$), or a sphere of radius $k^{-\frac{1}{2}}$ with its length metric (if $k > 0$), or the hyperbolic plane of curvature k (if $k < 0$). A *k -comparison triangle* for $[a, b, c]$ is a triangle $[\bar{a}, \bar{b}, \bar{c}]$ in the k -plane such that $|\bar{a}\bar{b}| = |ab|$, $|\bar{a}\bar{c}| = |ac|$ and $|\bar{b}\bar{c}| = |bc|$. A length space X “has curvature $\leq k$ ” if locally (i.e. in some neighborhood of each point of X) the following holds:

TRIANGLE CONDITION: For each geodesic triangle $[a, b, c]$ inside X and for any point d in the relative interior of $[a, b]$, one has $|cd| \leq |\bar{c}\bar{d}|$, where $[\bar{a}, \bar{b}, \bar{c}]$ is the k -comparison triangle for $[a, b, c]$ and \bar{d} is the unique point on $[\bar{a}, \bar{b}]$ with $|\bar{a}\bar{d}| = |ad|$.

Non-positively curved is synonymous with “of curvature ≤ 0 ”.

A CAT(k) space is a length space of curvature $\leq k$ in which the triangle condition holds globally. Sometimes CAT(0) spaces are called “spaces of non-positive curvature in the large”. Obviously, any CAT(0) space has curvature ≤ 0 . The converse is false: The cylinder $S^1 \times [0, 1]$ is non-positively curved (because it is locally isometric to $\mathbb{R} \times \mathbb{R}_{\geq 0}$) but not CAT(0), as shown by any geodesic triangle with all vertices on the circle $S^1 \times \{0\}$. Also, all CAT(0) spaces are contractible, while the cylinder is not even simply connected. This is not a coincidence, as explained by the Hadamard–Cartan theorem, which provides a crucial local-to-global correspondence:

Theorem 2.1 (Hadamard–Cartan, Gromov, Alexander–Bishop [2]). *Let X be any complete space with curvature ≤ 0 . The following are equivalent:*

- (1) X is simply connected;
- (2) X is contractible;
- (3) X is CAT(0).

In a CAT(0) space, any two points are connected by a unique geodesic. The same holds for CAT(k) spaces ($k > 0$), as long as the two points are at distance $< \pi k^{-\frac{1}{2}}$. However, while all CAT(0) spaces are contractible, some CAT(1) spaces are not simply connected.

Let A be a subset of a metric space X . A is called *convex* if any two points of A are connected by some shortest path in X that lies entirely in A . A is called *strongly convex* if there is a unique shortest path between any two points of A , and it lies in A . Obviously, strongly convex implies convex; in a CAT(0) space, the two notions are equivalent.

Let c be a point of X and let M be a subset of X , not necessarily convex. We denote by $\pi_c(M)$ the subset of the points of M at minimum distance from c . In case $\pi_c(M)$ contains a single point, with abuse of notation we write $\pi_c(M) = x$ instead of $\pi_c(M) = \{x\}$. This is always the case when M is convex, as the following, well known Lemma shows.

Lemma 2.2. *Let X be a connected CAT(k)-space. Let c be a point of X such that the function “distance from c ”, which maps any $x \in X$ into $|cx| \in \mathbb{R}$, has a unique local minimum on each convex subset A of X . In case k is positive and A lies in distance less than $\frac{\pi}{2}k^{-\frac{1}{2}}$ from c , $|cx|$ attains a unique local minimum in the disk of radius $\frac{\pi}{2}k^{-\frac{1}{2}}$ around c .*

Proof. Suppose two distinct points a, b in A are both local minima for the distance from c . Consider the geodesic triangle $[a, b, c]$. In the comparison triangle $\bar{a}\bar{b}\bar{c}$, one of the angles $\angle \bar{a}\bar{b}\bar{c}$ and $\angle \bar{b}\bar{a}\bar{c}$ must be acute. Let us assume $\angle \bar{b}\bar{a}\bar{c}$ is acute. (The other case is analogous.) For any point \bar{d} lying on $[\bar{a}, \bar{b}]$ sufficiently close to \bar{a} , one has $|\bar{c}\bar{d}| < |\bar{a}\bar{c}|$. Thus there is a neighborhood J_0 of a in X so that for any d_0 in J_0 the corresponding point \bar{d}_0 in the k -plane satisfies $|\bar{c}\bar{d}_0| < |\bar{a}\bar{c}|$. By the convexity assumption, there is a neighbourhood J_1 of a in X such that, for any point $d_1 \in J_1$ on the segment $[a, b]$, the shortest path $[a, d_1]$ lies entirely in A . Since in $\text{CAT}(k)$ spaces the geodesics are unique, $[a, d_1]$ is contained in $[a, b]$. By the local minimum assumption, there is a neighborhood J_2 of a in X so that $|ac| \leq |d_2c|$ for each $d_2 \in J_1 \cap A$. In conclusion, let d be a point close to a in $J_0 \cap J_1 \cap J_2$. Since $d \in J_0$, we have $|\bar{c}\bar{d}| < |\bar{a}\bar{c}|$. Since $d \in J_1 \cap J_2$, the geodesic $[a, d]$ is contained in A and so $|ac| \leq |cd|$. But by construction $|\bar{a}\bar{c}| = |ac|$, and by the triangle condition $|cd| \leq |\bar{c}\bar{d}|$; so $|\bar{c}\bar{d}| < |\bar{a}\bar{c}| = |ac| \leq |cd| \leq |\bar{c}\bar{d}|$, a contradiction. \square

2.2 Spaces of curvature bounded below and Cheeger's theorem

Let X be a (complete) length space and k a real number. We say that X has curvature $\geq k$ if for any geodesic triangle $[a, b, c]$ inside X and for any point d in the relative interior of $[a, b]$, one has $|cd| \geq |\bar{c}\bar{d}|$, where $[\bar{a}, \bar{b}, \bar{c}]$ is the k -comparison triangle for $[a, b, c]$ and \bar{d} is the unique point on $[\bar{a}, \bar{b}]$ with $|\bar{a}\bar{d}| = |ad|$. By Topogonov's theorem, it suffices to check the condition above locally. Topogonov's theorem can be viewed as an analogue of the Hadamard–Cartan theorem, but it does not require simply-connectedness among the assumptions, cf. [18].

Let k, D, v be real numbers, with $-\infty < k < \infty$ and $D, v > 0$. We say that a compact length space X satisfies the Cheeger constraints (k, D, v) if

- (i) the curvature of X is at least k ,
- (ii) the Hausdorff volume of X is at least v , and
- (iii) the diameter of X is at most D .

Spaces of curvature bounded below are “almost” Riemannian manifolds. In fact, by Otsu-Shioya's theorem, every space of curvature $\geq k$ must have a $C^{\frac{1}{2}}$ -Riemannian metric almost everywhere [65]. A theorem by Nikolaev says that if X has bounded curvature not only from below, but from above as well, then the interior of X can be uniformly approximated (in the sense of the Gromov-Hausdorff distance) by Riemannian manifolds of the same dimension, and with curvature bounds converging to the curvature bounds of X [18, p. 404]. Riemannian d -manifolds with Cheeger constraints (k, D, v) cannot contain short closed geodesics. In fact, for each triple $k, D, v \in \mathbb{R}$ and for any $d \in \mathbb{N}$, there is a constant C such that no Riemannian d -manifold with Cheeger constraints (k, D, v) has a closed geodesic shorter than C [20]. Manifolds with Cheeger constraints are not so many:

Theorem 2.3 (Cheeger [21]). *Let $-\infty < k \leq K < \infty$ and $D, v > 0$. There is a finite number of diffeomorphism types of compact Riemannian d -manifolds without boundary that satisfy the Cheeger constraints (k, D, v) and have also curvature $\leq K$.*

More recently, Grove, Petersen and Wu showed that when $d \geq 5$ the assumption of curvature $\leq K$ above can be dropped [45].

2.3 Simplicial complexes, geometric triangulations and paths

An *abstract simplicial complex* is a finite non-empty collection of subsets (“faces”) of $\{1, \dots, n\}$ that is closed under taking subsets. The dimension of a face is the number of its elements, minus one. The *dimension* of an abstract complex is the maximal face dimension.

If d is a positive integer, a d -*simplex* is the convex hull of $d+1$ generic points in \mathbb{R}^d . (“Generic” means that their affine hull is full-dimensional.) Similarly, a d -*simplex in S^d* is the convex hull, with respect to the standard intrinsic metric of S^d , of $d+1$ generic points of S^d . A *geometric simplicial complex* is a finite, nonempty collection of simplices in some \mathbb{R}^k or S^k , such that any two of them intersect in a common face. With abuse of notation, all the simplices in a geometric simplicial complex are called *faces*. The *underlying space* $|C|$ of C is the union of all its faces, as topological space. The *dimension* of C is the maximal dimension of a simplex in C .

Given any geometric simplicial complex C , let us label its vertices of C by $1, \dots, n$ and let us identify each face of C with the list of vertices contained in it; it is clear that the set of all faces of C forms an abstract simplicial complex A . We say that C is a *geometric realization* of A (or a *linear embedding* of A). Every abstract simplicial complex of dimension d admits some geometric realizations in \mathbb{R}^k (and in S^k), for some $k \in \{d, \dots, 2d+1\}$. Two geometric realizations of the same abstract complex are said to be “*combinatorially equivalent*”. It is easy to show that if C and C' are combinatorially equivalent, then C and C' are homeomorphic. Following a frequent convention in literature, we will use the expression *d -dimensional simplicial complexes* (or shortly “ *d -complexes*”) for geometric simplicial complexes of dimension d , modulo combinatorial equivalence; or equivalently, for abstract simplicial complexes of dimension d , modulo vertex relabelings. The notion of “underlying space of a complex” is well-defined up to homeomorphism.

The inclusion-maximal faces of a complex are called *facets*. A simplicial complex is *pure* if all facets have the same dimension. The *dual graph* of a pure simplicial complex C is the graph whose nodes correspond to the facets of C ; two nodes are connected by an arc if and only if the corresponding facets share a codimension-one face. A complex is called *strongly connected* if its dual graph is connected. If the underlying space $|C|$ of C is homeomorphic to a manifold M , we say that the complex C is a *triangulation* of M . Any triangulation of a connected manifold is pure and strongly connected. More generally, if M is a metric length space, we say that C is a *geometric triangulation* of M if there exists a metric μ on $|C|$ such that every k -simplex of C is a convex k -ball with respect to μ , and $(|C|, \mu)$ is isometric to M . Any geometric triangulation of a Riemannian manifold M is also a triangulation of M , but the converse is false. For example, Edwards’ non-PL triangulation of S^5 is not geometric [33].

The *join* $\sigma * \tau$ of two simplices σ and τ is a simplex whose vertices are the vertices of σ plus the vertices of τ . By convention, $\emptyset * \tau$ is τ itself. The *join* of two simplicial complexes S and T is defined as $S * T := \{\sigma * \tau : \sigma \in S, \tau \in T\}$. If σ and τ are two nonempty faces of a simplicial complex C , the *simplicial difference* $\sigma -_s \tau$ (also known as *deletion of τ from σ*) is the minimal face of C containing all vertices of σ that are not in τ . The *star* of σ , denote by $\text{star } \sigma$, is the subcomplex formed by all facets of C containing σ . The *link* of σ , denoted by $\text{link } \sigma$, is the subcomplex of C consisting of all faces of $\text{star } \sigma$ disjoint from σ . Clearly, $\text{star } \sigma = \sigma * \text{link } \sigma$. The *star boundary* of σ , denoted by $\text{starbd } \sigma$, is the subcomplex of all faces of $\text{star } \sigma$ that do not contain σ . This notation is justified by the fact that whenever C is the triangulation of a topological manifold and σ is an interior face of C , the “star boundary” of σ is indeed the boundary of $\text{star } \sigma$. Also, in case σ is a vertex, the star boundary coincides with the link.

The next two Lemmas are well-known; we include a short proof for the sake of completeness.

Lemma 2.4. *Let σ_1, σ_2 be two faces of a simplicial complex C . Let $\gamma : [0, 1] \rightarrow |C|$ be a continuous curve. Suppose that γ intersects the relative interior of σ_i at time t_i ($i = 1, 2$), and in addition γ intersects no other face of C in the interval (t_1, t_2) . Then one of the σ_i is contained in the other.*

Proof. Obviously the intersection $\sigma_1 \cap \sigma_2$ is non-empty. Since γ intersects no other face in the time interval, either $\sigma_1 \cap \sigma_2 = \sigma_1$ or $\sigma_1 \cap \sigma_2 = \sigma_2$. \square

Lemma 2.5. *Let C be a simplicial complex, $\dim C \geq 2$. Let γ be a path in the underlying space $|C|$, connecting a point x in the relative interior of a face σ to a point y in the relative interior of a face τ . Then, either $\sigma \subset \tau$, or γ contains a point that lies in the relative interior of some face of $\text{starbd } \sigma$.*

Proof. Let $\sigma = \sigma_1, \sigma_2, \dots, \sigma_k = \tau$ be the list of all the faces of C whose relative interior is “hit” by γ , in the order of appearance from x_1 to x_2 . Let j be the largest integer such that σ_j contains σ . If $j = k$, then $\tau \subset \sigma$ and we are done. If $j < k$, by Lemma 2.4 the face σ_{j+1} has to be contained in σ_j , which contains σ . By definition, σ_{j+1} belongs to $\text{starbd } \sigma$. \square

Discrete Morse theory, collapsibility and non-evasiveness

The *face poset* (C, \subseteq) of a complex C is the set of faces of C , ordered with respect to inclusion. By (\mathbb{R}, \leq) we denote the poset of real numbers with the usual ordering. A *discrete Morse function* is an order-preserving map f from (C, \subseteq) to (\mathbb{R}, \leq) , such that the preimage $f^{-1}(r)$ of any number r consists of at most 2 elements. We also assume that whenever $f(\sigma) = f(\tau)$, one of σ and τ is contained in the other. A *critical cell* of C is a face at which f is strictly increasing.

The function f induces a perfect matching on the non-critical cells: two cells are matched whenever they have identical image under f . This is called *Morse matching*, and it is usually represented by a system of arrows: Whenever $\sigma \subsetneq \tau$ and $f(\sigma) = f(\tau)$, one draws an arrow from the barycenter of σ to the barycenter of τ . We consider two discrete Morse functions *equivalent* if they induce the same Morse matching. Since any Morse matching pairs together faces of different dimensions, we can always represent a Morse matching by its “associated function” $\Theta : C \rightarrow C$, defined as follows. For each face σ of C ,

$$\Theta(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is unmatched,} \\ \tau & \text{if } \sigma \text{ is matched with } \tau \text{ and } \dim \sigma < \dim \tau. \end{cases}$$

A *discrete vector field* V on a simplicial complex C is a collection of pairs (σ, Σ) such that σ is a codimension-one face of Σ , and no face of C belongs to two different pairs of V . A *gradient path* in V is a concatenation of pairs of V

$$(\sigma_0, \Sigma_0), (\sigma_1, \Sigma_1), \dots, (\sigma_k, \Sigma_k),$$

so that σ_{i+1} is a codimension-one face of Σ_i for each i . This gradient path is called *closed* if $\sigma_0 = \sigma_k$ for some k . A discrete vector field V is a Morse matching if and only if V contains no closed gradient paths [37]. The main result of discrete Morse Theory is the following:

Theorem 2.6 (Forman [36, Theorem 2.2]). *Let C be a simplicial complex. Given any Morse matching on the face poset of C , the complex C is homotopy equivalent to a CW complex with one i -cell for each critical i -simplex.*

An *elementary collapse* is the simultaneous removal from a simplicial complex C of a pair of faces (σ, Σ) , such that Σ is the only face of C that properly contains σ . If $C' = C - \sigma - \Sigma$, we say that C *collapses* onto C' . We also say that the complex C *collapses onto* the complex D if C can be reduced to D by a finite sequence of elementary collapses. A *collapsible* complex is a complex that collapses onto a single vertex, or equivalently, a complex that admits a discrete Morse function with only one critical vertex and with no critical cells of higher dimension. Collapsible complexes are contractible; collapsible PL manifolds are necessarily balls [80]. However, some PL 3-balls are not collapsible [15] and some collapsible 6-balls (for example, the cones over non-PL 5-balls) are not PL.

Non-evasiveness is a further strengthening of collapsibility, emerged in theoretical computer science [50]. A 0-dimensional complex is non-evasive if and only if is a point. Recursively, a

d -dimensional simplicial complex ($d > 0$) is non-evasive if and only if there is some vertex v whose link and deletion are both non-evasive. The barycentric subdivision of every collapsible complex is non-evasive [78].

2.4 Rocket shellings, vertex decomposability and the Hirsch Conjecture

Let C be any pure simplicial d -complex with N facets. C is called *shellable* if either $d(N-1) = 0$, or $d(N-1) \neq 0$ and C splits as $C = C_1 \cup C_2$, where

- (i) C_1 is a shellable d -complex,
- (ii) C_2 is a d -simplex, and
- (iii) $C_1 \cap C_2$ is a shellable $(d-1)$ -complex.

Intuitively, shellable complexes can be assembled one d -face at the time. This induces a total order, called *shelling order*, on the facets of the complex. Shellable contractible complexes are collapsible, because we can collapse away all the top-dimensional faces in the reverse of the shelling order. As for the converse, a wedge of two triangles is collapsible, but not shellable.

In 1971, Bruggesser and Mani proved that the boundary of any convex simplicial $(d+1)$ -polytope is shellable. The crucial idea, as described in [82, pp. 239–245], is to imagine a rocket lifting off from one of the facets and flying away on a generic straight line l . The order in which the rocket sees the facets appear on the horizon, one after the other, is a shelling order for roughly half of the polytope (its “upper boundary”). This can be completed to a shelling of the full polytope by thinking of another rocket that lifts off from the opposite side of polytope, again along the same line l , but in the opposite direction [82].

Example 2.7 (Rudin’s ball and a non-polytopal sphere). Let v be a vertex of an arbitrary 4-polytope P . The deletion of v from ∂P yields a shellable 3-complex: To see this, choose a generic line of \mathbb{R}^4 through v , and perform a rocket shelling of ∂P . (The star of v is shelled first, the deletion is shelled next.) In 1958, Mary Rudin found a simplicial complex R with a convex realization in \mathbb{R}^3 , and with the following property: The removal from R of any tetrahedron destroys the ball topology [70]. In particular, R cannot be shellable, because no face can be the last one in a shelling. Consider the 3-sphere $S = \partial(v * R)$, where v is a new vertex. Were S combinatorially equivalent to the boundary of a 4-polytope P , R would be combinatorially equivalent to the deletion of a vertex from ∂P , and thus shellable: A contradiction. So S is not polytopal, although it is a geometric triangulation of S^3 .

Vertex decomposability was defined by Provan–Billera [67] as follows. Let C be any pure simplicial d -complex with N facets. C is called *vertex decomposable* if either $d(N-1) = 0$, or $d(N-1) \neq 0$ and there is a vertex v whose link is a vertex decomposable $(d-1)$ -complex and whose deletion is a vertex decomposable d -complex. The barycentric subdivision of any shellable complex is vertex decomposable [67], and every vertex-decomposable ball is shellable. All vertex decomposable contractible complexes are non-evasive, but the converse is false, as shown by a wedge of triangles. Even some 3-balls are non-evasive, but not vertex decomposable [12].

A strongly connected pure d -complex with n vertices is called *Hirsch* if the diameter of its dual graph is bounded above by $n - d - 1$. The *Hirsch conjecture* claims that the boundary of every simplicial $(d+1)$ -polytope is Hirsch. In 1978, Walkup constructed a non-polytopal 27-sphere that is not Hirsch [76]. The Hirsch conjecture was very recently settled by Santos, who found a high-dimensional counterexample. The *polynomial Hirsch conjecture*, which claims that the dual graph of any d -polytope with n vertices has diameter bounded above by a polynomial $P(n, d)$, is still a major open problem in combinatorics. Provan and Billera showed that vertex decomposable complexes are Hirsch [67]. In particular, the barycentric subdivisions of all polytope boundaries are Hirsch.

2.5 Acute and nonobtuse triangulations

Given two hyperplanes $\sum_{i=1}^d a_i x_i = 0$ and $\sum_{i=1}^d b_i x_i = 0$ in \mathbb{R}^d , the *dihedral angle* between them is the unique number θ in the interval $[0, \pi) \subset \mathbb{R}$ such that

$$\cos \theta = \left(\sum_{i=1}^d a_i b_i \right) \left(\sum_{i=1}^d a_i^2 \right)^{-\frac{1}{2}} \left(\sum_{i=1}^d b_i^2 \right)^{-\frac{1}{2}}.$$

A simplex is called *acute* (resp. *nonobtuse*) if the dihedral angle between the affine hulls of any two facets is smaller than $\frac{\pi}{2}$ (resp. smaller or equal than $\frac{\pi}{2}$). In any acute simplex, all faces are themselves acute simplices. In particular, all triangles in an acute simplex are acute in the classical sense. The same holds for nonobtuse simplices. A simplex is called *equilateral* if all edges have the same length. Obviously, all equilateral simplices are acute and all acute simplices are nonobtuse. The next lemma characterizes these notions in terms of orthogonal projections.

Lemma 2.8. *Let $|\Delta|$ be a geometric realization of a d -simplex Δ in \mathbb{R}^d , or, more generally, of a d -polytope Δ in some space of constant curvature. Then $|\Delta|$ is acute (resp. nonobtuse) if and only if, for each facet F of $\partial\Delta$, there is an orthogonal projection of $|\Delta|$ to the totally geodesic subspace spanned by $|F|$ such that $|\Delta| - |F|$ projects to the relative interior of $|F|$ (resp. to $|F|$).*

The Lemma above is easy to prove; a more general statement is proven in [1].

In 2004, Eppstein–Sullivan–Üngör showed that \mathbb{R}^3 can be tiled into acute tetrahedra [35]. This was later strengthened by Van der Zee et al. [75] and Kopczyński–Pak–Przytycki [54], who proved that the unit cube in \mathbb{R}^3 can be tiled into acute tetrahedra. In contrast, there is no geometric triangulation of the 4-cube into acute 4-simplices [54]. For $d \geq 5$, neither \mathbb{R}^d nor the $(d+1)$ -cube have acute triangulations, cf. [54]. In contrast, by subdividing a cubical grid, one can obtain nonobtuse triangulations of \mathbb{R}^d and of the d -cube for any d . So, acute is a much more restrictive condition than nonobtuse.

Let C be a simplicial complex. We can define a metric structure on the underlying space $|C|$ by realizing each simplex in some space of constant curvature c , and then by gluing together the realizations according to the combinatorics of C . A simple way to do this is to assign the same length to all edges: This induces a piecewise Euclidean metric on C , which corresponds to realizing all simplices of C as the “standard” simplex in the space of constant curvature 0. If this metric turns C into a CAT(0) length space, we say that the complex C is CAT(0) *with the equilateral flat metric*.

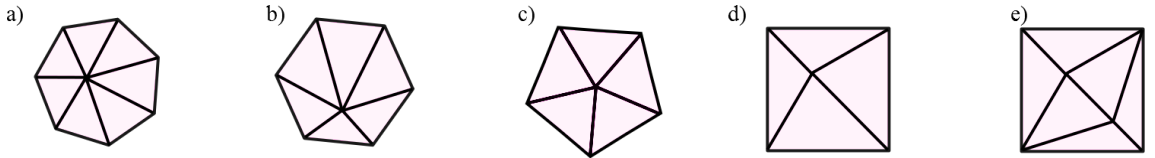


Figure 1: Complex (a) is CAT(−1) with an equilateral hyperbolic metric. Complex (b) is CAT(0) with the equilateral flat metric. Complex (c) is not CAT(0) with the equilateral flat metric; it is CAT(0) with the acute metric that assigns length 1 resp. 1.4 to all interior resp. boundary edges, say. Complex (d) is flat with the non-obtuse metric that assigns length 1 resp. $\sqrt{2}$ to all interior resp. boundary edges. Finally, only an obtuse metric can make complex (e) CAT(0); but the complex is CAT(1) with a nonobtuse metric.

More generally, given a complex C and a real number k , C is called CAT(k) *with an acute* (resp. *nonobtuse*) *metric* if one can choose the realization of the simplices in some space of constant curvature k such that all simplices are acute (resp. nonobtuse) and the resulting metric on $|C|$ is CAT(k).

3 Main Results

3.1 Gradient Matchings and Star-Minimal Functions

Here we obtain non-trivial Morse matchings on a simplicial (or polytopal) complex, by studying real-valued continuous functions on some geometric realization of the complex.

Definition 3.1 (Star-minimal). Let C be a simplicial complex, and let $|C|$ be the underlying space of C . A function $f : |C| \rightarrow \mathbb{R}$ is called *star-minimal* if it satisfies the following three conditions:

- (i) f is continuous,
- (ii) on the star of each vertex, f has a unique absolute minimum, and
- (iii) no two vertices have the same value under f .

Condition (iii) is just a technical detail, since it can be forced by ‘wiggling’ the complex a bit, or by perturbing f . Some embedded complexes may be ‘resistant to wiggling’, cf. e. g. [28]; in this case, we can still perform a ‘virtual wiggling’ by choosing a total order \triangleleft on the vertices of C which extends the partial order given by f ; so that if $f(x) \leq f(y)$, then $x \triangleleft y$.

Conditions (i) and (ii) are also not very restrictive, as the next result shows.

Proposition 3.2. *Every embedding of every pure simplicial complex admits some star-minimal function.*

Proof. Choose an embedding $|C|$ of C . Let S be the $(d - 1)$ -skeleton of C , and let $|S|$ be the corresponding subspace of $|C|$. Any continuous function that vanishes on $|S|$ and has negative values on all points of $|C| \setminus |S|$, is star-minimal. \square

Our next goal is to show that any star-minimal function on a complex C naturally induces a certain type of Morse matching, called *gradient matching*. The key is to define a “pointer function” $y_f : C \rightarrow C$, which intuitively maps each face into the “best” vertex of its star. (How good a vertex is, is decided simply by looking at its image under f .) Unlike f , which is defined on $|C|$, the map y_f is purely combinatorial. Later, we will obtain a matching from y_f basically by pairing together every not-yet-matched face σ with the face $\sigma * y_f(\sigma)$.

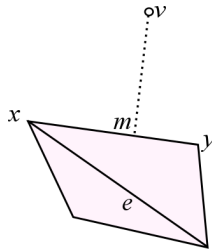


Figure 2: How to match an edge e with respect to the function $f = \text{distance from some point } v$. The function has a unique minimum m on $|\text{star } e|$. The inclusion-minimal face containing m in its interior is spanned by the vertices x and y . Between these two vertices, we choose the one with minimal distance from v . If it is x , since x is contained in e , we do nothing. If it is y , which does not belong to e , we match e with $e * y$.

In details: Let σ be a face of C . On the star of σ , which is the intersection of the stars of its vertices, f has a unique minimum m . We denote by $\mu(\sigma)$ the inclusion-minimal face among all the faces of $\text{star}(\sigma)$ that contain the point m . We set

$$y_f(\sigma) = \mu(\sigma),$$

where y is the f -minimal vertex of $\mu(\sigma)$. Since \triangleleft is a total order, the pointer function y_f is well-defined. (By convention, we set $y_f(\emptyset) := \emptyset$.)

Next, we define a matching $\Theta_f : C \longrightarrow C$ associated with the function f . The definition is recursive on the dimension: In other words, we start defining Θ_f for the faces of lower dimension, working all the way up to the facets. Let σ be a face of C . Set $\Theta_f(\emptyset) = \emptyset$. If for all faces $\tau \subset \sigma$ with $\dim \tau < \dim \sigma$ one has $\Theta_f(\tau) \neq \sigma$, we define

$$\Theta_f(\sigma) := y_f(\sigma) * \sigma.$$

Note that every face of C is either in the image of Θ_f , or in its domain, or both. In the latter case, $\Theta_f(\sigma) = \sigma$.

Definition 3.3 (Gradient matching). Let $\Theta : C \longrightarrow C$ be (the function associated to) a matching on the faces of a simplicial complex C . We say that Θ is a *gradient matching* if

$$\Theta = \Theta_f,$$

for some star-minimal function $f : |C| \rightarrow \mathbb{R}$, and for some embedding $|C|$ of C .

Remark 3.4. Our definition implicitly uses the “pointer function” y_f that we defined above. However, the pointer function can also be defined differently. The fact that y minimizes f is not so relevant in the definition of the map y_f . One might as well choose $y_f(\sigma)$ to be any vertex in $\mu(\sigma)$, as long as the choice depends only on μ .

Next we show that all gradient matchings are indeed Morse matchings. Moreover, they have “relatively few” critical cells. This gives us the first geometric method valid in all dimensions to construct efficient Morse matchings.

Theorem 3.5. *Let C be a simplicial complex, embedded in some \mathbb{R}^k . Let $f : |C| \rightarrow \mathbb{R}$ be any star-minimal continuous function on the underlying space of C . Then the induced gradient matching Θ_f is a Morse matching. Moreover, the map*

$$\sigma \mapsto (y_f(\sigma), \sigma)$$

yields a bijection between the set \mathfrak{C} of the (nonempty) critical faces and the set

$$\mathfrak{P} := \{(v, \tau) \mid v \in \tau, y_f(\tau) = v \text{ and } y_f(\tau -_s v) \neq v\}.$$

In particular, C admits a discrete Morse function with c_i critical i -simplices, where

$$c_i = \# \{(v, \tau) \mid v \in \tau, \dim \tau = i, y_f(\tau) = v \text{ and } y_f(\tau -_s v) \neq v\}.$$

Proof. Our proof has four parts:

- (I) the pairs $(\tau, \Theta_f(\tau))$ form a discrete vector field;
- (II) this discrete vector field contains no closed gradient path;
- (III) the map $\sigma \mapsto (y_f(\sigma), \sigma)$ from \mathfrak{C} to \mathfrak{P} is well-defined;
- (IV) the map $\sigma \mapsto (y_f(\sigma), \sigma)$ is a bijection.

Part I. *The pairs $(\tau, \Theta_f(\tau))$ form a discrete vector field.* We have to show that any face appears in at most one pair $(\tau, \Theta_f(\tau))$. In other words, we have to show that Θ_f is injective. Suppose Θ_f maps two distinct k -faces σ_1, σ_2 to the same $(k+1)$ -face Σ . By definition of Θ_f ,

$$y_f(\sigma_1) * \sigma_1 = \Sigma = y_f(\sigma_2) * \sigma_2.$$

All vertices of Σ are contained either in σ_1 or in σ_2 . So $\text{star } \sigma_1 \cap \text{star } \sigma_2 = \text{star } \Sigma$. Moreover, each $y_f(\sigma_i)$ belongs to Σ but not to σ_i ; so $y_f(\sigma_i)$ must belong to σ_{3-i} . This means that the function f attains its minimum on $\text{star } \sigma_i$ at a point m_i which lies in σ_{3-i} . Since on $\text{star } \sigma_1 \cap \text{star } \sigma_2 = \text{star } \Sigma$ f has a unique minimum, we obtain $m_1 = m_2$. But then

$$y_f(\sigma_1) = y_f(\sigma_2) = y_f(\Sigma).$$

Since $y_f(\sigma_1) * \sigma_1 = y_f(\sigma_2) * \sigma_2$, it follows that $\sigma_1 = \sigma_2$, a contradiction.

Part II. *The discrete vector field contains no closed gradient path.* Consider the function $\min_{\text{star } \sigma} f$ from $|C|$ to \mathbb{R} . Choose an arbitrary gradient path $\sigma_1, \dots, \sigma_m$. In this path, consider a pair σ_i, σ_{i+1} with $\dim \sigma_i < \dim \sigma_{i+1}$. By definition of gradient path, σ_i and σ_{i+1} are matched with one another, σ_{i+2} is matched with σ_{i+3} and so on. In particular $\Theta_f(\sigma_i) = \sigma_{i+1}$ and $\Theta_f(\sigma_{i+2}) = \sigma_{i+3}$. Then $\min_{\text{star } \sigma_i} f = \min_{\text{star } \sigma_{i+1}} f$. We claim that in this case

$$\min_{\text{star } \sigma_{i+1}} f > \min_{\text{star } \sigma_{i+2}} f.$$

In fact, suppose by contradiction that $\min_{\text{star } \sigma_{i+1}} f = \min_{\text{star } \sigma_{i+2}} f$. Note that $\sigma_{i+2} \subset \sigma_{i+1}$. By the star-minimality of f , the local minimum of f must be unique. So

$$y_f(\sigma_i) = y_f(\sigma_{i+1}) = y_f(\sigma_{i+2}).$$

So $y_f(\sigma_{i+2})$ is the unique vertex in σ_{i+1} but not in σ_i . In particular, $y_f(\sigma_{i+2})$ belongs to σ_{i+2} , so $\Theta_f(\sigma_{i+2}) = \sigma_{i+2}$ and σ_{i+2} is a critical cell. Yet this contradicts the fact that σ_{i+2} appears in a gradient path. In conclusion, for any gradient path $\sigma_1, \dots, \sigma_m$ one has $\min_{\text{star } \sigma_i} f < \min_{\text{star } \sigma_{i+2}} f$. This excludes the possibility that $\sigma_{2k+1} = \sigma_1$ for some k .

Part III. *The map $\sigma \mapsto (y_f(\sigma), \sigma)$ from \mathfrak{C} to \mathfrak{P} is well-defined.* Recall that

$$\mathfrak{P} := \{(v, \tau) \mid v \in \tau, \ y_f(\tau) = v \text{ and } y_f(\tau -_s v) \neq v\}.$$

Consider an arbitrary vertex w of C . Either $y_f(w) \neq w$, or $y_f(w) = w$. If $y_f(w)$ is a vertex x different than w , then w is in the domain of Θ_f ; moreover, $\Theta_f(w)$ is the edge $[x, w]$ and w is not critical. If $y_f(w) = w$, again w is in the domain of Θ_f ; one has $\Theta_f(w) = w$, so w is critical. In this case it is easy to verify that (w, w) belongs to \mathfrak{P} .

Now, consider a critical face σ of dimension $k \geq 1$. Since σ is critical, there is no $(k-1)$ -face τ such that $\Theta_f(\tau) = \sigma$. By the definition of Θ_f , σ is in the domain of Θ_f ; so since σ is critical we must have $\Theta_f(\sigma) = \sigma$. So the vertex $y_f(\sigma)$ belongs to σ . In order to conclude that $(y_f(\sigma), \sigma) \in \mathfrak{P}$, we only have to prove that $y_f(\sigma -_s y_f(\sigma)) \neq y_f(\sigma)$. Let us adopt the shortenings $v := y_f(\sigma)$, $\delta := \sigma -_s v$. Suppose by contradiction that $y_f(\delta) = v$. If δ were not in the image of any of its facets under the map Θ_f , then we would have $\Theta_f(\delta) = \sigma$, which would contradict the assumption that σ is a critical face. So, there has to be a codimension-one face ρ of δ such that $\Theta_f(\rho) = \delta$. In other words, $y_f(\rho)$ is the unique vertex that belongs to δ but not to ρ . Clearly $\arg\min_{\text{star } \rho} f \in \text{star } y_f(\rho)$, whence

$$\arg\min_{\text{star } \rho} f = \arg\min_{\text{star } (\rho * y_f(\rho))} f = \arg\min_{\text{star } \delta} f.$$

So, $y_f(\rho) = y_f(\delta)$. Recall that we are assuming $y_f(\delta) = v$, where $\delta := \sigma -_s v$. Hence

$$y_f(\rho) = y_f(\delta) = y_f(\sigma -_s v) = v \notin \sigma -_s v.$$

This contradicts the fact that $\Theta_f(\rho) = \sigma -_s v$. Thus, the assumption $y_f(\delta) = v$ must be wrong.

Part IV. The map $\sigma \mapsto (y_f(\sigma), \sigma)$ from \mathfrak{C} to \mathfrak{P} is a bijection.

The map $s : \sigma \mapsto (y_f(\sigma), \sigma)$ is clearly injective. Let us verify surjectivity. Consider a pair (v, τ) in \mathfrak{P} . By definition of s , $y_f(\tau) = v$ and $y_f(\tau -_s v) \neq v$. Assume that $s(\tau) \notin \mathfrak{P}$, or, equivalently, that τ is not critical.

Since $y_f(\tau) = v \in \tau$, there must be a facet η of τ such that $\Theta_f(\eta) = \tau$. By definition of Θ_f , $\operatorname{argmin}_{\operatorname{star} \eta} \in \operatorname{star} \tau$, hence $\operatorname{argmin}_{\operatorname{star} \eta} = \operatorname{argmin}_{\operatorname{star} \tau}$. Since y_f only depends on the point where f attains the minimum, $y_f(\tau) = y_f(\eta) = y_f(\tau -_s v) = v$. This contradicts the assumption that (v, τ) is in \mathfrak{P} . Thus, τ is critical, as desired. \square

Remark 3.6. Theorem 3.5 implies that every gradient matching is a Morse matching. In fact, one can give a complete characterization of gradient matchings within Morse matchings, as follows: A Morse Matching is a gradient matching if and only if for every matching pair $(\sigma, \Theta(\sigma))$ there exists a facet $\Sigma \supset \Theta(\sigma)$ of C such that, for any face τ of Σ that contains σ but does not contain $\Theta(\sigma)$, we have

$$\Theta(\tau) = \tau * \Theta(\sigma).$$

Remark 3.7. Not all collapsible complexes are collapsible via a gradient matching. To see this, consider the collapsible 2-complex C described in [8, Example 5.4]: This complex has the property that all collapsing sequences for C start by removing the same edge e . Let us attach to a single tetrahedron three copies of C ; each copy should be glued by identifying e with some edge of the tetrahedron. The resulting 3-complex D is collapsible, so it admits a matching which leaves only one vertex unmatched. However, it is easy to see that any gradient matching on D leaves several faces unmatched.

3.2 Discrete Hadamard–Cartan Theorem

In this Subsection, we prove our discrete version of the Hadamard–Cartan theorem, namely, that CAT(0) complexes with convex vertex links are simplicially collapsible (**Theorem 3.8**). This strengthens a result by Crowley, which we briefly introduce.

Crowley’s approach uses the classical notion of *minimal disk*, the disk of minimum area spanned by a 1-sphere in a simply connected complex. Gromov [43] and Gersten [39, 40] have studied minimal disks in connection with group presentations and the word problem; later Chepoi and others [7, 23, 24] have used them to relate CAT(0) complexes with median graphs. To collapse a complex onto a fixed vertex v , Crowley’s idea is to reduce the problem to the two-dimensional case, by studying the minimal disk(s) spanned by two geodesics that converge to v . Her argument is based on two observations:

- (1) If the complex is three-dimensional, these minimal disks are CAT(0) with the equilateral flat metric, as long as the starting complex is. This was proven in [29, Theorem 2] and independently rediscovered by Chepoi and Osajda [24, Claim 1]; see also [47, 48].
- (2) Any contractible 3-dimensional pseudomanifold is collapsible if and only if it admits a discrete Morse function without critical edges.

None of these two facts extends to higher-dimensional manifolds:

- (1) If we take two consecutive cones over the pentagon of Figure 1, the resulting complex is obviously collapsible, but not with Crowley’s argument: Some minimal disk in it contains a degree-five vertex.
- (2) A priori, a contractible d -manifold different than a ball ($d \geq 6$) could admit a discrete Morse function with 1 critical vertex, m critical $(d - 3)$ -faces, m critical $(d - 2)$ -faces, and no further critical face. Compare the smooth result by Sharko [73, pp. 27–28].

Our new approach consists in applying the ideas of Section 3.1 to the case where C is a CAT(0) complex, and the function $f : |C| \rightarrow \mathbb{R}$ is the distance from a given vertex v of C . Because of

the CAT(0) property, this function is star-minimal — so it will induce a convenient gradient matching on the complex. The main advantage of this new approach is that it works in all dimensions.

Theorem 3.8. *Let C be a simplicial complex embedded in some \mathbb{R}^k . Suppose its underlying space $|C|$ is endowed with a metric such that*

- (i) *for each vertex x in C , the underlying space of star x in $|C|$ is geodesically convex, and*
- (ii) *either $k \leq 0$ and $|C|$ is CAT(k), or $k > 0$ and $|C|$ is CAT(k) with diameter $\leq \frac{\pi}{2\sqrt{k}}$.*

Then C is collapsible.

Proof. Fix a vertex w of C . Let $f : |C| \rightarrow \mathbb{R}$ be the function

$$f(x) = |xw|,$$

which maps each point of $|C|$ into its distance from w . Let us perform on the face poset of C the Morse matching constructed in Theorem 3.5. Clearly, the vertex w will be mapped onto itself. Moreover, by Lemma 2.4, for each vertex $u \neq w$ we have $\operatorname{argmin}_{\operatorname{star}(u)} f \in \operatorname{starbd}(w)$, whence $y_f(u) \neq u$. So every vertex is matched with an edge, apart from w , which is the only critical vertex.

By contradiction, suppose there is a critical face τ of dimension ≥ 1 . Set $v := y_f(\tau)$. On star τ , the function f attains its minimal value in the relative interior of a face $\sigma_v \subset \operatorname{star} \tau$ that contains v . Let δ be any face of star $\tau -_s v$ containing σ_v . Clearly, δ contains v . Thus, star τ and star $(\tau -_s v)$ coincide in a neighborhood of v . By Lemma 2.2, $\operatorname{argmin}_{\operatorname{star}(\tau -_s v)} f = \operatorname{argmin}_{\operatorname{star} \tau} f$. Therefore $y_f(\tau -_s v) = y_f(\tau) = v$. This means that

$$(v, \tau) \notin \{(v, \tau) | v \in \tau, y_f(\tau) = v \text{ and } y_f(\tau -_s v) \neq v\};$$

hence by Theorem 3.5 τ is not critical, a contradiction. \square

Remark 3.9. In any pure CAT(0) space, every locally convex subset in which every two points are connected by a rectifiable path is also convex. So in the assumptions of Theorem 3.8 it suffices to check condition (ii) locally: We can replace condition (ii) with the request that star σ is (geodesically) convex for every ridge σ . In fact, if all ridges have convex stars, the nearest-point projection to star (v) is a well-defined map and it is locally non-expansive for every vertex v . But then the projection to star (v) is also globally non-expansive. Now let x and y be any two points in star (v) , and let us assume the geodesic γ from x to y leaves star (v) . Let us project γ to star (v) . The result is a curve γ' connecting x and y that is obviously lying in star (v) , and not longer than γ . This contradicts the uniqueness of shortest paths in CAT(0) spaces.

Corollary 3.10. *Let C be a simplicial complex. Suppose that $|C|$ is CAT(0) with a nonobtuse metric. Then C is collapsible.*

Proof. By the assumption, there is a metric structure on $|C|$ of non-positive curvature, such that every face of C is nonobtuse. In nonobtuse triangulations, the star of every ridge is convex. In fact, let Σ, Σ' be two facets containing a common ridge R . Since Σ and Σ' are convex, and their union is locally convex in a neighborhood of R , $\Sigma \cup \Sigma'$ is locally convex. Since the embedding space is CAT(0), convexity follows as in Remark 3.9. \square

Corollary 3.11. *Every complex that is CAT(0) with the equilateral flat metric, is collapsible.*

Corollary 3.12 (Crowley [29]). *Every 3-pseudomanifold that is CAT(0) with the equilateral flat metric, is collapsible.*

We conclude the section proving that Theorem 3.8 can be extended to polytopal complexes. The key for this is Bruggesser–Mani’s rocket shelling of polytope boundaries.

Definition 3.13 (Polytopal complex). A *polytope* in \mathbb{R}^k (resp. S^k) is the convex hull of finitely many point in \mathbb{R}^k , endowed with the Euclidean metric (resp. S^k , endowed with the standard intrinsic metric). A *polytopal complex* in \mathbb{R}^k (resp. S^k) is a finite, nonempty collection of polytopes in \mathbb{R}^k (resp. S^k), such that any two of them intersect in a common face.

Definition 3.14 (Polytopal join, difference). Let P be a facet of a polytopal complex C , and let σ and τ be two disjoint faces of P . The *join* $\sigma * \tau$ is the minimal polytope of C that contains both σ and τ . If S is any subcomplex of C , the *difference* $C -_r S$ is the subcomplex of all the faces of C that belong to some facet not in S .

Lemma 3.15. *For any polytope P and for any face σ , P collapses polyhedrally onto $\text{star}(\sigma, \partial P)$.*

Proof. Performing a rocket shelling of ∂P with a generic line through σ , one gets a shelling of ∂P in which $\text{star}(\sigma, \partial P)$ is shelled first [82, Cor. 8.13]. Let τ be the last facet of such shelling. Any contractible shellable complex is collapsible; the collapsing sequence of the facets is given by the inverse shelling order. Therefore, $\partial P - \tau$ collapses onto $\text{star}(\sigma, \partial P)$. But as a polytopal complex, P trivially collapses polyhedrally onto $\partial P - \tau$; so P collapses also onto $\text{star}(\sigma, \partial P)$. \square

Theorem 3.16. *Let C be a polyhedral complex. Suppose we can endow its underlying space $|C|$ with a length metric such that*

- (i) *for each vertex x in C , the underlying space of $\text{star } x$ in $|C|$ is geodesically convex, and*
- (ii) *either $k \leq 0$ and $|C|$ is CAT(k), or $k > 0$ and $|C|$ is CAT(k) with diameter $\leq \frac{\pi}{2\sqrt{k}}$.*

Then C is collapsible.

Proof. Fix a vertex v and an embedding $|C|$ of C . Let $f^C : |C| \rightarrow \mathbb{R}$ be the distance from v in $|C|$. This f^C is a function that has a unique local minimum on the star of each cell.

We prove the claim by induction. Let σ be a facet of C maximizing $\min_{|\sigma|} f^C$ in a minimal face μ , and let $F \in \text{star } \mu$ be the complex of faces that attain their minimum at μ . Any two facets of F intersect in a face containing μ because of the uniqueness of minima. By Lemma 3.15, we can collapse the facets of F to

$$\text{star } \mu \cap \bigcup_{\tau \subset F} \partial \tau.$$

In particular, we can collapse the complex C to the complex C' of all faces of C that are contained in a facet of C that is not in F .

We still have to show the existence of a function f on C' that attains a unique local minimum on each vertex star. Let us show that the restriction $f_{|C'}^C$ of f^C to C' is the function we are looking for. Assume that for some vertex w of C the function $f_{|C'}$ attains two local minima on $\text{star}(w, C')$. Let x be the absolute minimum of f restricted to $\text{star}(w, C')$; let y be the other (local) minimum. Let τ be a facet containing y . When restricted to C , f attains a unique local minimum on the star of every face. Therefore, the point y must lie in $F \cup C$. But y is not the local minimum of F on M , and thus, $\min_{|\tau|} f^C$ is larger than $\min_{|F|} f^C$, contradicting the way the complex F was defined. \square

In particular, Theorem 3.16 holds for “cubical CAT(0) complexes”, which are complexes of regular unit cubes glued together to yield a CAT(0) metric space. These complexes have been extensively studied in literature: See e.g. [5], [14], [30], [43]. One example of cubical CAT(0) complex is the space of phylogenetic trees, introduced by Billera, Holmes and Vogtmann [14]:

Corollary 3.17. *The space of phylogenetic trees is collapsible.*

Remark 3.18. Discrete Morse Theory works also in the broader generality of regular CW-complexes. However, Theorem 3.8 does not extend beyond the world of polytopal complexes. To see this, choose any simplicial d -sphere S that does not collapse after the removal of any facet. (For how to construct one such example, see Lickorish [56].) Let P be the (strongly) regular CW complex obtained by attaching a $(d + 1)$ -cell Σ to S , along the whole boundary of Σ . Clearly, P is homeomorphic to a ball, so it admits a CAT(0) metric. Also, all vertex stars in P coincide with P itself and are thus convex. However, as a regular cell complex P does not collapse: Any collapsing sequence should start by removing the $(d + 1)$ -cell Σ together with some d -face σ , yet the resulting d -complex $S - \sigma$ is not collapsible, by the way S was chosen.

3.3 Discrete Cheeger Theorem

For the convergence of certain discrete quantum gravity models, it is of crucial interest to prove exponential bounds for the number of triangulations of manifolds [4]. The counts are always performed in terms of the number N of facets, and for fixed dimension; two triangulations are considered equal if they have isomorphic face posets.

Here we prove, under certain convexity assumptions, that there are only exponentially many combinatorial types of Riemannian manifolds with bounded volume, fixed dimension, and with a metric of with upper and lower curvature bounds. (**Theorem 3.27**). Since arbitrary manifolds are more than exponentially many, our result can be viewed as a discrete analogue of the Cheeger theorem, which bounds the number of diffeomorphism types of manifolds with bounded curvature, diameter and volume (**Theorem 2.3**). The main idea of our proof is to cover the manifold with a bounded number of balls, and then to use discrete Morse theory, especially the techniques established in [11].

To make the reader more familiar with the techniques of counting combinatorial types of manifolds asymptotically, let us start with a couple of examples. The first one shows that if we want to reach exponential bounds for triangulations of d -manifolds, and d is at least two, we *must* add some geometric assumption.

Example 3.19. Starting with a $1 \times 4g$ grid of squares, let us triangulate the first $2g$ squares (left to right) by inserting “backslash” diagonals, and the other $2g$ squares by “slash” diagonals. If we cut away the last triangle, we obtain a triangulated disc B with $8g - 1$ triangles. This B contains $2g$ disjoint triangles. In fact, if we set

$$a_j := \begin{cases} 4j - 2 & \text{if } j \in \{1, \dots, g\} \\ 4j - 1 & \text{if } j \in \{g + 1, \dots, 2g\} \end{cases}$$

the triangles in position a_1, \dots, a_{2g} (left to right) are disjoint. For simplicity, we relabel these $2g$ disjoint triangles by $1, \dots, g, 1', \dots, g'$.

Given a new vertex v , we form the cone $S_B := \partial(v * B)$ and remove from it the interiors of the triangles $1, \dots, g, 1', \dots, g'$. The resulting “sphere with $2g$ holes” can be completed to a closed surface by attaching g handles. More precisely, let us fix a bijection $\pi : \{1, \dots, g\} \rightarrow \{1', \dots, g'\}$. In the triangle i , let us denote by x_i resp. u_i the leftmost resp. the upper vertex; symmetrically, in the triangle $\pi(i)$ let us call $x_{\pi(i)}$ resp. $u_{\pi(i)}$ the rightmost resp. the upper vertex. For each $i \in \{1, \dots, g\}$, we can attach a (non-twisted) triangular prism onto the holes i and $\pi(i)$, so that x_i resp. u_i gets connected via an edge to $x_{\pi(i)}$ resp. $u_{\pi(i)}$. Each prism can be triangulated with six facets by subdividing each lateral rectangle into two; as a result, we obtain a simplicial closed 2-manifold $M_g(\pi)$.

The number of triangles of $M_g(\pi)$ equals $16g$ (the number of triangles of S_B) minus $2g$ (the holes) plus $6g$ (the handles). Therefore, $M_g(\pi)$ has genus g and $20g$ facets. Any two different

permutations π and ρ give rise to two combinatorially different surfaces $M_g(\pi)$ and $M_g(\rho)$. Thus, there are at least $g!$ surfaces with genus g and $20g$ triangles. For N large, $\frac{N}{20}!$ grows faster than any exponential function. So, surfaces are more than exponentially many.

Example 3.20. Let us subdivide both the top and bottom edge of a unit square S into r segments of equal length. By triangulating S linearly without adding further vertices into $N = 2r$ triangles, we obtain roughly

$$C_{N+2} = \frac{1}{N+3} \binom{2N+4}{N+2}$$

different triangulations (without accounting for symmetry) of S , where C_{N+2} is the $(N+2)$ -nd Catalan Number. For N large, this number grows exponentially in N : In fact, Stirling's formula yields the asymptotics

$$C_n \approx \frac{4^n}{n^{1.5}\sqrt{\pi}}.$$

By identifying the sides of the square to a torus, we get roughly 4^N combinatorially distinct triangulations of the torus with N facets. By passing to a 2-fold cover, we can also ensure that the stars of vertices are all (strongly) convex.

We leave it to the reader to conclude the following Lemma:

Lemma 3.21. *Let M be a space form of dimension $d \geq 2$. There are at least exponentially many triangulations of M in which all vertex stars are strongly convex.*

This motivates the search for an upper bound to the number of such geometric triangulations. We will show that Lemma 3.21 is best possible, in the sense that these triangulations are also at most exponentially many (Theorem 3.27).

First we recall the notion of *endo-collapsibility*, introduced in [11]. A triangulation C of a d -manifold with non-empty (resp. empty) boundary is called *endo-collapsible* if C minus a d -face collapses onto ∂C (resp. onto a point). This notion is interesting for us for the following two reasons:

Theorem 3.22 (Benedetti–Ziegler [13], [11, Theorem 3.6]). *For fixed d , in terms of the number N of facets, there are at most exponentially many triangulations of endocollapsible d -manifolds with N facets.*

Lemma 3.23 (Benedetti [11, Corollary 3.21]). *Let B be a collapsible PL d -ball. If $\text{sd link } \sigma$ is endocollapsible for every face σ , then $\text{sd } B$ is endocollapsible.*

Not all geometric triangulations are endo-collapsible. However, Theorem 3.8 provided us a tool to claim that complexes satisfying certain metric assumptions are collapsible. Now, via Lemma 3.23, we are able to conclude that their barycentric subdivisions are endo-collapsible.

Lemma 3.24. *Let T be either a geometric triangulation of the standard S^d , or a geometric triangulation of a convex polytope in S^d . If all vertex stars of T are convex in S^d , then $\text{sd } T$ is endo-collapsible.*

Proof. We prove the lemma by induction on the dimension. For $d = 2$, all 2-spheres and 2-balls are shellable and in particular endo-collapsible.

Let P be any geometric triangulation of a convex polytope in S^d with convex vertex stars. For every nonempty face σ of P , $\text{link } \sigma$ is a geometric triangulation of a manifold of dimension lower than d : Specifically, a sphere if σ is interior, or a ball if σ is in the boundary of P . In both

cases, link σ is a geometric triangulation with convex vertex stars. By the inductive assumption, $\text{sd link } \sigma$ is endo-collapsible. Furthermore, P is collapsible by Theorem 3.8. By Lemma 3.23, $\text{sd } P$ is endo-collapsible.

Let T be a geometric triangulation of the standard S^d with convex vertex stars. Let H^+ be a hemisphere of S^d . We can choose H^+ to be generic, so that the boundary of H^+ contains no face of T . Let T^+ be the subcomplex of faces of T which lie strictly in H^+ . Let U be the subcomplex of faces of T that intersect $S^d \setminus H^+$. By Theorem 3.8, both T^+ and U are collapsible. Furthermore, for any nonempty interior face σ of U , link σ is a geometric triangulation of a sphere with smaller dimension than d . (If instead σ is in ∂U , link σ is a geometric triangulation of a convex polytope in a sphere of dimension less than d). All vertex stars in link σ are convex. By induction, $\text{sd } U$ is endocollapsible. We still have to show that $\text{sd } T$ is endocollapsible. If we remove a facet Δ from $\text{sd } U \supset \text{sd } T$, by the endo-collapsibility of $\text{sd } U$ we can collapse $\text{sd } T$ onto $\text{sd } \partial U = \text{sd } (U \cap T^+)$. The “leftover” L of $\text{sd } T$ is combinatorially equivalent to $\text{sd } T^+$. Since T^+ is collapsible, so is $\text{sd } T^+$. Thus, L collapses to a point. In particular, $\text{sd } T$ minus a facet collapses onto a point. \square

All the ingredients are there to discretize the finiteness theorem by Cheeger (Theorem 2.3). The idea is now to chop a triangulation of manifold with bounded geometry, into a bounded number of endocollapsible balls. The key for this is given by the following two lemmas: One is Cheeger’s bound on the injectivity radius, the other a direct consequence of Topogonov’s theorem.

Lemma 3.25 (Cheeger [20, 22]). *Let $-\infty < k < \infty$ and $D, v > 0$. There exists a positive number $C = C(k, D, v)$ such that every Riemannian d -manifold with Cheeger constraints (k, D, v) has no closed geodesic of length less than C .*

Lemma 3.26. *Let k, D be real numbers, with $D > 0$. Let d be a positive integer. For every $\varepsilon > 0$, there exists a number N_ε such that every Riemannian d -manifold with curvature bounded below by k and diameter at most D can be covered with at most N_ε balls of radius ε .*

Proof. Let X be a Riemannian manifold satisfying the assumptions. Let x be a point of X . Let B_D^d be a ball of radius D in the d -dimensional space of constant curvature $m = \min\{k, 0\}$. B_D^d has a cover with N balls of diameter ε . Consider the map $\text{e}\tilde{\text{x}}\text{p}^{-1} \exp_x$, where $\text{e}\tilde{\text{x}}\text{p}$ is the exponential map in B_D^d with respect to the center and \exp_x is the exponential map in X with respect to x . Compare [20, Chapter 1, Section 2]. The domain of this map is set to be the preimage of X . By Topogonov’s Theorem, X has curvature $\geq k$. It is not difficult to see that this implies that for each $a, b \in B_D^d$, $|\text{e}\tilde{\text{x}}\text{p}^{-1} \exp_x(a) \text{e}\tilde{\text{x}}\text{p}^{-1} \exp_x(b)| \leq |ab|$; in other words, $\text{e}\tilde{\text{x}}\text{p}^{-1} \exp_x$ is a non-expansive map. Thus, the images of the N_ε balls that cover B_D^d are contained in N balls of radius at most ε . \square

Theorem 3.27. *For fixed k, K, D and v , in terms of the number N of facets, there are only exponentially many geometric triangulation with strongly convex vertex stars of d -dimensional Riemannian manifolds of curvature bounded below by k and bounded above by K , of diameter $\leq D$ and of volume $\geq v$.*

Proof. Our proof has three parts:

- (i) we cover a Riemannian manifold X satisfying the constraints above with strongly convex open balls;
- (ii) we count the number of geometric triangulations restricted to each ball;
- (iii) we assemble the triangulated balls together, thus estimating the number of triangulations of X .

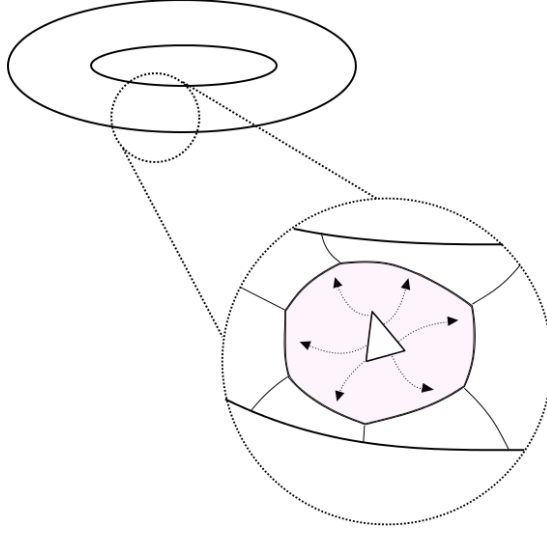


Figure 3: The restriction of the triangulation to each of the A_i ($i \in S$) is endo-collapsible. By counting the number of ways in which two of the A_i 's can be glued to one another, we determine an upper bound on the number of triangulations.

Part (i). Let X be a Riemannian manifold of dimension d satisfying the Cheeger constraints (k, D, v) , with curvature bounded above by K .

By Lemma 3.26, there exists a set S of $s = s(k, K, d, v)$ points in such that every point in X lies in distance less than $\varepsilon = \min\{\frac{C(k, D, v)}{4}, \frac{\pi}{4\sqrt{K}}\}$ of S . By the Hadamard–Cartan theorem, and its extension by Ballmann [6, Theorem 7], any ball of radius ε in X is $\text{CAT}(K)$. Let T be a triangulation of X into N simplices such that vertex stars are strongly convex. Let $(A_i)_{i \in S}$ be the family of open convex balls with radius ε , centered at the points of S . By the way ε was chosen, the A_i are also strongly convex, because they are $\text{CAT}(K)$.

Part (ii). For any subset $A \subset X$, we denote by T_A the complex of faces of T intersecting A . Let A_i be one of the convex balls that cover X . T_{A_i} is collapsible by Theorem 3.8. Also, for every face σ of T_{A_i} , $\text{sd link } \sigma$ is endo-collapsible by Lemma 3.24. Thus, by Lemma 3.23, $\text{sd } T_{A_i}$ is endo-collapsible. Since every facet of T intersects A_i at most once, by Theorem 3.22, there is a constant C such that the number of combinatorial types of T_{A_i} is bounded above by e^{CN} .

Part (iii). The triangulation T of X is completely determined by

- (i) the triangulation of each T_{A_i} ,
- (ii) the triangulation $T_{A_i \cap A_j}$ and its position in T_{A_i} and T_{A_j} . (This means we have to specify which of the faces of T_{A_i} are faces of $T_{A_i \cap A_j}$ too. The same holds for the faces of T_{A_j} .)

As we saw in Part II, we have e^{CN} choices for triangulating each T_{A_i} . Since $T_{A_i \cap A_j}$ is a subcomplex of both T_{A_i} and T_{A_j} , we have 2^N choices for every $T_{A_i \cap A_j}$, with $(i, j) \in \binom{S}{2}$. Since $T_{A_i \cap A_j}$ is strongly connected, it suffices to determine the location of one facet Δ of $T_{A_i \cap A_j}$ in $T_{A_i \cap A_j}$ and T_{A_i} (including its orientation) to determine the position of $T_{A_i \cap A_j}$ in T_{A_i} . For this, we have at most $(d+1)!^2 N^2$ possibilities. Thus the number of triangulations T with strongly convex vertex stars of d -dimensional Riemannian manifolds with N facets, diameter $\leq D$, volume $\geq v$, and curvature between k and K , is bounded above by

$$e^{CsN} 2^{\binom{s}{2}N} ((d+1)!N)^{2s(s-1)}.$$

Since $s = s(k, K, D, v)$ does not depend on the number of facets N , this is the desired exponential bound. \square

Remark 3.28. Let us compare our discrete Cheeger theorem with the original setting of discrete quantum gravity. In the dynamical triangulations model, which uses the equilateral flat metric, the volume of any triangulated d -manifold with N facets is simply N times the volume of the regular d -simplex. Up to rescaling the edge length, the volume can thus be identified with the number of facets; this convention is adopted often, in the discrete quantum gravity literature. In Theorem 3.27 we are *not* following this convention. In fact, our model is more general, because we consider any nonobtuse metric, and not just the equilateral one. (In some sense, this is the spirit of Regge calculus, rather than that of Weingarten’s model.) So, even if we put restrictions on the volume, it makes perfect sense to let N grow to infinity, and to count the number of triangulations asymptotically with respect to N .

We should mention that a metric approach à-la-Cheeger to the dynamical triangulations model has been attempted as early as 1996, by four Italian mathematical physicists [9]. Unfortunately, the paper [9] is according to the authors themselves “incomplete in many respects” [9, p. 257]. In particular, it is not completely clear which triangulations are actually counted (namely, which ones arise as nerves of ball coverings). However, our Theorem 3.27 achieves a more general statement with a rigorous metric–combinatorial proof, valid in all dimensions.

3.4 Complexes with convex geometric realizations

In dimension $d \leq 3$, all convex d -balls are collapsible, whether their vertex stars are convex or not. This allows us to simplify/extend the proof of Theorem 3.27. The result is a bound on the number of lower-dimensional *space forms*, which are Riemannian manifolds with uniform sectional curvature.

Theorem 3.29. *For fixed k, D and v , in terms of the number N of facets, there are only exponentially many geometric triangulations of space forms with curvature $\geq k$, diameter $\leq D$, volume $\geq v$, and dimension at most 3.*

Proof. The proof is the same as for Theorem 3.27, except that instead of Theorem 3.8, we use the fact that every convex 3-ball is collapsible and endo-collapsible [13, 25]. The same is true a fortiori for 2-balls, which are all shellable. \square

Corollary 3.30. *In terms of the number of facets, there are only exponentially many geometric triangulations of the standard S^3 .*

Since we can straighten any triangulation of a surface until we obtain a triangulation into convex simplices [27], Corollary 3.30 can be viewed as a 3D-version of the old result by Tutte that polytopal 2-spheres are exponentially many [74]. In higher dimension, d -polytopes are still exponentially many: This leads us to the natural question of whether there are only exponentially many geometric triangulations of the d -sphere. There is however a major difficulty: The collapsibility of convex d -balls has been proven only in dimension $d \leq 3$ [25]. Chillingworth and Lickorish asked whether every linear triangulation of a d -simplex is collapsible; more generally, Goodrick asked whether d -complexes with star-shaped linear embeddings are collapsible. Both questions are open; compare Kirby [52, Problem 5.5].

Here we show that *any* linear subdivision of the d -simplex becomes collapsible, after a constant number of barycentric subdivisions (Theorem 3.42). The good news is, the constant is universal: It depends only on the dimension d , but not on the triangulation L . Let us recall some definitions first.

Definition 3.31 (Star-shaped). A subspace $A \subset \mathbb{R}^d$ is called *star-shaped* if there exists a point x in A , called a *center of A* , such that for each y in A the straight line segment $[x, y]$ lies entirely

in A . Similarly, a subspace $B \subset S^d$ is called *star-shaped* if B lies in a closed hemisphere of S^d , and there exists a point x in B , in the interior of the hemisphere, such that for each y in B the geodesic from x to y lies entirely in B . With abuse of notation, a polytopal complex T (in \mathbb{R}^k or in S^k) is called star-shaped if some geometric realization of T is star-shaped.

Definition 3.32 (First order neighborhood). Let T be a polytopal complex. Let P be a subcomplex of T . The *first order neighborhood* $N^1(P)$ of P is the polytopal complex formed by the faces of $\text{sd } T$ intersecting $\text{sd } P$ in $\text{sd } T$.

The following lemma is well known.

Lemma 3.33. *Let T be a polytopal complex in \mathbb{R}^k (resp. in S^k). Let $N(T)$ be a set of points in \mathbb{R}^k (resp. in S^k), one in the relative interior of each face of T . There exists a realization of $\text{sd } T$ with $N(T)$ as vertex set.*

Lemma 3.34. *Let T be a star-shaped polytopal complex in a closed hemisphere h of S^k . Let P be the subcomplex of faces of T that do not intersect the boundary of h , and assume that any face σ of T in ∂h is the facet of a unique face τ of T whose relative interior lies in the interior of h . Then the first order neighborhood $N^1(P)$ has a star-shaped geometric realization in \mathbb{R}^k .*

Proof. Let m be the midpoint of h . Let $B_r(m)$ be the closed metric ball in h with midpoint m and radius r . If there is some $r < \frac{\pi}{2}$ such that $T \subset B_r(m)$, then T has a realization as a star-shaped set in \mathbb{R}^d by central projection, and we are done. Thus, we can assume that T intersects ∂h . Let x be a center of T . Since $|P| \cup \{x\}$ is a compact subset of the relative interior of h , there is some open interval $J = (R, \frac{\pi}{2})$, contained in the interval $(0, \frac{\pi}{2}]$, such that if r is in J , the ball $B_r(m)$ contains both x and the whole of P .

If σ is a face of T in ∂h , let v_σ be any point in the relative interior of σ . If τ is a face of T intersecting P , but not contained in P , define $\sigma(\tau) := \tau \cap \partial h$. This is again a face of T . For each $r < \frac{\pi}{2}$ in the interval J , choose a point $w(\tau, r)$ in the relative interior of $B_r(m) \cap \tau$, so that the $w(\tau, r)$ depend continuously on R and tend to $v_{\sigma(\tau)}$, as r approaches $\frac{\pi}{2}$. Extend each family $w(\tau, r)$ continuously to $r = \frac{\pi}{2}$, by defining $w(\tau, \frac{\pi}{2}) = v_{\sigma(\tau)}$.

Next, we use this one-parameter family of points to produce a one-parameter family N_r of geometric realizations of $N^1(P)$, which is the simplicial complex corresponding to chains Θ of faces of T whose minimal element τ intersects P . The vertices of N_r should be in 1–1 correspondence with the faces of T intersecting P . So, let τ be a face of T intersecting P . If τ is in P , let x_τ be any point in the relative interior of τ . If τ is not in P , let $y_\tau = w(\tau, r)$. $N^1(P)$ is a subcomplex of $\text{sd } T$, and thus N_r is realizable as a simplicial complex with vertexset $w(\tau, r)$ by Lemma 3.33. Since $\sigma(\tau) = \sigma(\tau')$ if and only if $\tau = \tau'$, $v_{\sigma(\tau)} = \lim_{r \rightarrow \frac{\pi}{2}} w(\tau, r)$ is equal to $v_{\sigma(\tau')} = \lim_{r \rightarrow \frac{\pi}{2}} w(\tau', r)$ if and only if $\tau = \tau'$. Thus $\lim_{r \rightarrow \frac{\pi}{2}} N_r = N_{\frac{\pi}{2}} \cong N^1(P)$, since no two vertices of N_r converge to the same point when r tends to $\frac{\pi}{2}$.

Hence, N_r can be realized as a simplicial complex in $B_r(m) \subset h$. We claim that if r is close enough to $\frac{\pi}{2}$, then N_r is star-shaped with center x .

In fact, let us call *extremal faces* the faces of N_r all of whose vertices are in $\partial B_r(m)$. Let τ, τ' be extremal faces of N_r (possibly coinciding). We say that an ordered pair (τ, τ') of faces is *folded* if there are two distinct points a and a' , in the relative interior of τ resp. τ' , such that the ray from a through a' in $B_r(m)$ contains x , but the segment $[a, a']$ does not contain x . When $r = \frac{\pi}{2}$, folded faces do not exist, because all rays passing through interior points of τ and τ' lie strictly in $\partial B_{\frac{\pi}{2}}(m) = \partial h$. Because the vertices $w(\tau, r)$ of N_r depend continuously on r , we can find a real number $R' > R > 0$, such that for any r in the open interval

$$J' := \left(R', \frac{\pi}{2}\right) \subset J = \left(R, \frac{\pi}{2}\right) \subset \left(0, \frac{\pi}{2}\right),$$

the geometric complex N_r contains no folded pair of faces. For the rest of the proof, let us assume that $r \in J'$, so that folded pairs of faces are avoided. Let y be a point in $|N_r|$. We have to prove that the straight line segment $[x, y]$ lies in $|N_r|$. By contradiction, assume that $[x, y]$ leaves $|N_r|$. If $y \in N_r$ is not in an extremal face, there exists a open neighborhood U of y such that $U \cap |N_r(P)| = U \cap |T'|$. This means that only the extremal faces of N_r are (possibly) modified. In particular, $[x, y] \cap U \cap |N_r(P)| = [x, y] \cap U \cap |T'|$, so if $[x, y]$ leaves and reenters $|N_r|$, it must do so through extremal faces. Hence, there are two faces τ, τ' with vertices in $B_r(m)$ that intersect $[x, y]$ in vertices a, a' , but the interior of the segment from a to a' is not in $|N_r|$. In particular, $a \neq a'$. Without loss of generality, assume that the order of vertices along $[x, y]$ is x, a', a, y . Then, the ray of from a through a' intersects x . Thus τ, τ' is a folded pair of faces. But since r is in J' , there are no folded pairs of faces: A contradiction. Therefore, the claim is proven: $[x, y]$ lies in N_r , which is star-shaped in $B_r(m)$.

Let ϕ be a central projection that takes h to \mathbb{R}^k . The image $\phi(N_r) \subset \phi(B_r(m))$ is compact, so in particular bounded. Moreover, $\phi(N_r)$ is star-shaped in \mathbb{R}^k with center $\phi(x)$. Hence, $\phi(N_r)$ is the geometric realization of $N^1(P)$ we were looking for. \square

Next, we introduce “special” geometric realizations of the barycentric subdivision. Our goal is to make sure that the vertices of the barycentric subdivisions will be nicely ordered; later on, we will delete the vertices in this order to prove non-evasiveness.

Definition 3.35 (ν -directed subdivisions). Let d, k be integers, with $d \leq k$. Let P be a polytopal complex in \mathbb{R}^k , and let ν be a vector of \mathbb{R}^k . Assume that ν is generic, that is, no face of P is orthogonal to ν . Let v_τ denote the vertex of $\text{sd } P$ corresponding to the face τ of P . A ν -directed subdivision of P is a linear derived subdivision, where the new vertices are chosen such that, for any 2 faces $\sigma \subsetneq \tau$, either σ is the vertex of τ minimizing the internal product $\langle v_\sigma, \nu \rangle$, or

$$\langle v_\tau, \nu \rangle < \langle v_\sigma, \nu \rangle.$$

The existence of this subdivision follows directly from Lemma 3.33. The subdivision is not unique; it is unique up to combinatorial equivalence.

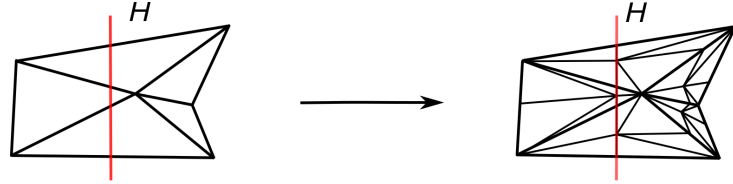


Figure 4: An example of H -splitting subdivision.

Definition 3.36 (H -splitting subdivisions). Let d, k be integers, with $d \leq k$. Let P be a polytopal complex in \mathbb{R}^k , and let H be a generic hyperplane of \mathbb{R}^k . Let ν be the vector orthogonal to H . A H -splitting subdivision of P is a linear derived subdivision, with vertices chosen so that:

- they form a ν -directed subdivision of any face of P that lies in the open halfspace bounded by H in direction ν ;
- they form a $(-\nu)$ -directed subdivision of any face of P that lies in the open halfspace bounded by H in direction $-\nu$;
- for any face τ of σ that intersects the hyperplane H , the vertex v_τ lies on H .

Definition 3.37 (Split link). Let T be a simplicial complex in \mathbb{R}^k , and let ν be a generic vector in \mathbb{R}^k , that is, no face of P is orthogonal to ν . Let v be a vertex of T . The *split link* (of T at v with respect to ν), denoted by $\text{Slink}^\nu(v, T)$, is the intersection of link (v, T) with the hemisphere h of directions in link (v, T) with midpoint $-\nu$.

If H^- is the halfspace in direction $-\nu$ delimited by $\nu^\perp + v$, then $\text{Slink}^\nu(v, T) \cong \text{link}(v, T \cap H^-)$. This $\text{Slink}^\nu(v, T)$ is a polytopal complex in a hemisphere h . If a face σ lies on the boundary ∂h of such hemisphere, it corresponds uniquely to a face τ of T intersected by ∂h : In fact, σ is the unique facet of $\tau \cap h$ that intersects ∂h .

Definition 3.38 (Lower link). Let T be a simplicial complex in \mathbb{R}^k , and let ν be a generic vector in \mathbb{R}^k , as above. The *lower link of T at v with respect to ν* , denoted by $\text{Llink}^\nu(v, T)$, is the subcomplex of $\text{link}(v, T)$ induced by vertices w of T with $\langle \nu, w \rangle < \langle \nu, v \rangle$. This $\text{Llink}^\nu(v, T)$ is naturally a subcomplex of $\text{Slink}^\nu(v, T)$. Specifically, $\text{Llink}^\nu(v, T)$ can be obtained from $\text{Slink}^\nu(v, T)$ by deleting all vertices of $\text{Slink}^\nu(v, T)$ that belong to ∂h .

Definition 3.39. A *non-evasiveness step* is the deletion from a simplicial complex C of a single vertex whose link is non-evasive. Given two simplicial complexes B and C , we write $B \searrow_{\text{NE}} C$ if there is a series of nonevasiveness steps which lead from B to C .

The following Lemmas are well known and easy to prove.

Lemma 3.40. If $B \searrow_{\text{NE}} C$, then $\text{sd}^m B \searrow_{\text{NE}} \text{sd}^m C$ for all m .

Lemma 3.41. Let C be a simplicial complex and v a new vertex. Let m be a positive integer. If $v * (\text{sd}^m C) \searrow_{\text{NE}} \text{sd}^m C$, then $\text{sd}^m(v * C) \searrow_{\text{NE}} \text{sd}^m C$.

Theorem 3.42. Let B be a simplicial complex of dimension $d \geq 2$. If some geometric realization of B in \mathbb{R}^d is star-shaped, then $\text{sd}^{d-2}(B)$ is collapsible and non-evasive.

Proof. We proceed by induction on the dimension. Without loss of generality, we can assume that the origin $(0, \dots, 0)$ is a center of $B \subset \mathbb{R}^d$. Let ν be a generic direction in \mathbb{R}^d , such that no edge of B is orthogonal to ν . Let H be the hyperplane through the origin orthogonal to ν . From now on, let $\text{sd} B$ denote the H -splitting subdivision of B . Since ν contains a center of B , $\text{sd} B \cap H$ is a linear triangulation of a star-shaped set. Let H^+ (resp. H^-) be the closed halfspace delimited by H in direction ν (resp. $-\nu$). We make the following four claims:

- (1) $\text{sd}^{d-3} \text{Llink}^\nu(v, \text{sd} B)$ is non-evasive.
- (2) $\text{sd}^{d-2} B \cap H^+ \searrow_{\text{NE}} \text{sd}^{d-2} B \cap H$.
- (3) $\text{sd}^{d-2} B \cap H^- \searrow_{\text{NE}} \text{sd}^{d-2} B \cap H$.
- (4) $\text{sd}^{d-2} B \cap H$ is non-evasive.

Once we prove these claims, we are done. In fact, the sequence of vertex deletions in (2) deforms $\text{sd}^{d-2} B$ into $\text{sd}^{d-2} B \cap H^-$. The sequence in (3) reduces the latter complex to $\text{sd}^{d-2} B \cap H$, which in turn is non-evasive by (4). So, let us show the claims in their order.

- (1) Let v be a vertex of B that lies in the interior of H^+ . Clearly, v is also a vertex of $\text{sd} B$. By Lemma 3.34, $N^1(\text{Llink}^\nu(v, B))$ has a star-shaped geometric realization in \mathbb{R}^{d-1} . By induction, $\text{sd}^{(d-1)-2} N^1(\text{Llink}^\nu(v, B))$ is nonevasive. Thus we only need to show

$$N^1(\text{Llink}^\nu(v, B)) \cong \text{Llink}^\nu(v, \text{sd} B).$$

The vertices w of $\text{sd} B$ with $\langle \nu, v \rangle > \langle \nu, w \rangle$ and that are adjacent to v are in 1–1 correspondence with faces of B that contain v , but whose minimal vertex with respect to $\langle \nu, - \rangle$ is not v . These are precisely the faces of $\text{link}(v, B)$ that intersect $|\text{Slink}^\nu(v, B)|$. In $\text{sd} B$, these faces give rise to the vertices w , which thus correspond precisely to the vertices of $N^1(\text{Llink}^\nu(v, B)) \cong \text{Llink}^\nu(v, \text{sd} B)$.

- (2) Consider the vertex v of $\text{sd} B$ maximizing the internal product $\langle \nu, - \rangle$. This v is necessarily a vertex of B . Also, $\text{Llink}^\nu(v, \text{sd} B) = \text{link}(v, \text{sd} B)$, and by item (1) we have that

$\text{sd}^{d-2} \text{Llink}^\nu(v, \text{sd } B)$ is nonevasive. So deleting the vertex v is a honest non-evasiveness step. Lemma 3.40 yields

$$\text{sd}^{d-3} B \searrow_{\text{NE}} \text{sd}^{d-2}(\text{sd } B - v).$$

Next, we select the vertex w of the remaining complex C that maximizes $\langle \nu, - \rangle$. Note that $\text{link}(w, C) \cong \text{Llink}^\nu(w, \text{sd } B)$, since all higher vertices have been removed already. There are two cases to consider:

- If w corresponds to a face τ of B of positive dimension, then the vertices of $\text{link}(w, C) \cong \text{Llink}^\nu(w, \text{sd } B)$ correspond to faces of B containing both τ and the vertex of τ minimizing $\langle \nu, - \rangle$. Thus, $\text{link}(w, C)$ is a cone over $\text{link}(\tau, B)$. Every cone is non-evasive. Thus, $C \searrow_{\text{NE}} C - w$. By Lemma 3.40, $\text{sd}^{d-2} C \searrow_{\text{NE}} \text{sd}^{d-2}(C - w)$.
- If w corresponds to a vertex of B , $\text{link}(w, C) = \text{Llink}^\nu(w, \text{sd } B)$. (All higher vertices have been removed already.) Since by claim (1) the $(d-3)$ -rd barycentric subdivision of $\text{Llink}^\nu(w, \text{sd } B)$ is nonevasive, by Lemma 3.41 we have that $\text{sd}^{d-2} C \searrow_{\text{NE}} \text{sd}^{d-2}(C - w)$.

We can proceed deleting vertices, until the remaining complex has no vertex in the interior of H^+ . Thus $\text{sd}^{d-2} B \cap H^+ \searrow_{\text{NE}} \text{sd}^{d-2} B \cap H$, as desired.

- (3) This is symmetric to (2).
- (4) This is an easy consequence of the inductive assumption. In fact, $\text{sd } B \cap H$ is star-shaped and has dimension $d-1$: So, by induction, the subdivision $\text{sd}^{(d-1)-2}(\text{sd } B \cap H)$ is non-evasive. From the definition of H -splitting subdivision, $\text{sd}^{d-3}(\text{sd } B \cap H) = (\text{sd}^{d-2} B) \cap H$. \square

Theorem 3.43. *Let B be a simplicial complex of dimension $d \geq 2$.*

- (i) *If some geometric realization of B is convex in \mathbb{R}^d , then $\text{sd}^{d-2}(B)$ is a collapsible and endo-collapsible ball.*
- (ii) *If some geometric realization of B is star-shaped in \mathbb{R}^d , and B is a PL manifold, then B is a ball and $\text{sd}^{d-2}(B)$ is endo-collapsible and collapsible.*
- (iii) *If S is a geometric triangulation of S^d , then $\text{sd}^{d-1}(S)$ is endocollapsible.*

Proof. (i) and (ii) can be proved analogously to the proof of Theorem 3.42, by induction on the dimension. The only additional fact we need is that, with the notation of the proof of Theorem 3.42, if B is a PL manifold and v lies in H^+ then $\text{Slink}^\nu(v, B)$ is a PL manifold.

The proof of (iii) is analogous to the proof of Lemma 3.24: First we divide S into two star-shaped subsets of S^d , by selecting a generic hyperplane in S^d that contains no vertex of S , and by realizing $\text{sd } S$ in such a way that each vertex of $\text{sd } S$ corresponding to a face σ of S that intersects H lies in H . (This is the analogous of the splitting subdivision for complexes in S^d .) By Lemma 3.34, both parts have a star-shaped geometric realization in \mathbb{R}^d . By (ii), both parts are endocollapsible after $d-2$ barycentric subdivisions. \square

Finally we are ready to extend Theorem 3.29 to all dimensions.

Theorem 3.44. *For fixed k, D and v and d , in terms of the number N of facets, there are only exponentially many geometric triangulations of space forms with curvature $\geq k$, diameter $\leq D$, volume $\geq v$, and dimension d .*

Proof. Instead of using Chillingworth's Theorem, we use Theorem 3.43. The rest is analogous to the proof of Theorems 3.29 and 3.27. \square

Corollary 3.45. *There are exponentially many geometric triangulations of the standard S^d .*

Corollary 3.46. *There are exponentially many triangulations of the d -ball with N facets that admit a star-shaped linear embedding in \mathbb{R}^d .*

4 Applications

4.1 Vertex-transitive triangulations

In this short section we show a connection, suggested to us by Anders Björner, between metric geometry and the evasiveness conjecture. A *vertex-transitive* complex is a simplicial complex with n vertices on which S^n acts transitively; roughly speaking, this means that all vertices look alike. An important open problem in theoretical computer science is whether there is any vertex-transitive non-evasive simplicial complex, apart from the simplex. This is known as *evasiveness conjecture* [50][58].

It is known that collapsibility is not enough to force a vertex-transitive complex to be a simplex [58]. However, non-evasiveness is strictly stronger than collapsibility. We have shown in Section 3.2 that the property of “being CAT(0) with the equilateral flat metric” is also strictly stronger than collapsibility. Thus it makes sense to compare it with vertex-transitivity, in parallel with the statement of the evasiveness conjecture. Here is the result:

Theorem 4.1. *Every vertex-transitive simplicial complex that is CAT(0) with the equilateral flat metric, is a simplex.*

Proof. Let C be a vertex-transitive complex and let v_1, \dots, v_n be the vertices of C . Let \mathcal{E} be the metric space obtained endowing C with the equilateral flat metric. Let $f : \mathcal{E} \rightarrow \mathbb{R}$ be the function defined as $f(x) = \sum_{i=1}^n d(x, v_i)$. When \mathcal{E} is CAT(0), the function f is convex and has a *unique* minimum m .

Since the triangulation is vertex transitive, and the function f is also equivariant under a permutation of the vertices, we claim that

$$d(m, v_1) = d(m, v_2) = \dots = d(m, v_n),$$

so that m minimizes simultaneously all functions $x \mapsto d(x, v_i)$. In fact, were $d(m, v_i) < d(m, v_j)$ for some $i \neq j$, we could find by symmetry a point m' such that

(1) $d(m', v_j) = d(m, v_i) < d(m, v_j) = d(m', v_i)$, and

(2) $d(m', v_k) = d(m, v_k)$, if $k \notin \{i, j\}$.

But then we would have $f(m) = f(m')$: A contradiction, f has a unique minimum.

So, there is one point in the complex equidistant from all of the vertices. This implies that \mathcal{E} has only one facet, because for each facet F of \mathcal{E} , the unique point equidistant from all vertices of F is the barycenter of F itself. \square

Proposition 4.2. *“CAT(0) with the equilateral flat metric” and “non-evasive” are independent strengthenings of the collapsibility property.*

Proof. Let B be the stellar subdivision of a d -simplex Δ , obtained by connecting the barycenter b of Δ with all the vertices of Δ . Clearly, B is non-evasive and collapsible. Also, B is flat and in particular CAT(0) with a suitable piecewise-flat obtuse metric. However, when endowed with an arbitrary non-obtuse metric, B is *not* CAT(0). Compare Figure 1.

Let C be a collapsible 2-complex in which none of the vertex links is acyclic. Many examples of this type are known, cf. e. g. [8, Example 5.4]. By definition, C cannot be non-evasive. Alas, [8, Example 5.4] allows a nonobtuse CAT0 metric as well, thus, Theorem 3.8 can not be strengthened to conclude non-evasiveness. \square

4.2 Flag manifolds are Hirsch

Recall that a pure simplicial complex is called *Hirsch* if the diameter of its dual graph is at most $n - d - 1$, where n is the number of vertices and d the dimension of the complex. In this section, we prove that certain complexes satisfying curvature bounds are Hirsch. The idea is fairly simple: Curvature bounds allow us to set up some sort of combinatorial distance within the complex, so that we can move from a facet “towards” another facet. In each step, we leave a vertex star behind. As a corollary, we obtain that all triangulated manifolds whose minimal non-faces are edges (the so-called “flag manifolds”) are Hirsch.

Following the terminology of Mani and Walkup, we say that a pure complex C is W_v if between any two facets of the complex there is a path in the dual graph of C that visits every vertex of C at most once. In other words, the path intersects the star of every vertex along a connected interval (possibly empty, or consisting of a single point). As noticed by Mani and Walkup, all W_v complexes are Hirsch [59].

Theorem 4.3. *Let k be a real number. Let C be a simplicial complex satisfying the following conditions:*

- (i) *all faces of C (except possibly ridges) have pure and strongly connected links;*
- (ii) *endowed with a metric that assigns to each simplex the metric of a non-obtuse simplex in the space of constant curvature k , C is a $\text{CAT}(k)$ metric space.*

Then C is W_v and in particular Hirsch.

Remark 4.4. If $|C|$ is a Riemannian manifold, then condition (ii) can be replaced by the weaker request that “for every ridge σ , with respect to a metric that makes C a $\text{CAT}(k)$ space, $|\text{star } \sigma|$ is geodesically convex”. However, if $|C|$ is an arbitrary simplicial complex, condition (ii) cannot be weakened (at least not with our proof below). The reason is that we are going to prove Theorem 4.3 by induction on the dimension, arguing that if σ is a face in a polyhedral $\text{CAT}(k)$ space, then $\text{link } \sigma$ is a polyhedral $\text{CAT}(1)$ space with nonobtuse simplices. This inductive step fails when focusing on complexes with convex vertex stars. In fact, let C be the complex formed by four Euclidean triangles glued along a vertex, such that the angles at the common vertex are all $\frac{\pi}{2}$, except for one, which is strictly larger than $\frac{\pi}{2}$. This C is a $\text{CAT}(0)$ complex with convex vertex stars. The link of the central vertex is $\text{CAT}(1)$; however, the vertex stars in such link are not convex.

Proof of Theorem 4.3. Let C be a d -complex with n vertices that satisfies the conditions (i) and (ii) above. First we want to show that C is Hirsch, that is, that the diameter of the dual graph of C is at most $n - d - 1$. The case $d = 1$ is easy: Any spanning tree of C has $n - 1$ edges, thus the diameter of the dual graph of C can be at most $n - 2$. So, let us proceed by induction on d .

Choose two facets X, Y of C , and let x, y be points in X and Y , respectively. Let γ be a shortest path connecting x and y , parametrized with unit speed from x to y . Let us define $f : C \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$f(\sigma) := \max \{ t \mid \gamma(t) \in \text{star } \sigma \} \cup \{-\infty\}.$$

We want to construct a dual path from X to Y . Suppose we have constructed a dual path until a facet F_i different than Y . There are two cases to consider:

- (1) If γ leaves F_i through the relative interior of a ridge R , denote by F_{i+1} the facet crossed by γ after R . (In particular, $F_{i+1} \cap F_i = R$.) The dual path can be incremented of one step, to include the facet F_{i+1} .
- (2) If instead γ leaves F_i through a face σ of dimension $\leq d - 2$, we consider the simplicial complex $\text{link } \sigma$. The path γ hits $\text{link } \sigma$ in two points: The ray coming from x corresponds



Figure 5: (a): How to get a combinatorial path from a shortest path γ . Since all vertex stars are convex, the path γ exits the star of any vertex at most once. (b): If the convexity assumption on vertex stars is removed, a shortest path γ may leave a single vertex star multiple times.

to a point γ_- lying in a facet G_- of link σ corresponding to F_i in C ; the ray towards y corresponds to a point $\gamma_+ \neq \gamma_-$ in link σ . Let η be a shortest path in link σ from γ_- to γ_+ , and let G_+ be the first facet of the link crossed by η that contains γ_- . The simplices in link σ are nonobtuse, and by Gromov's criterion [43, Theorem 4.2.A], the link of σ is CAT(1). By induction, we can apply the procedure to find a dual path in link σ connecting G_- to G_+ . Let us lift the path to C , by joining its facets with σ . This allows us to increment the original dual path of several steps, until we reach the facet $\sigma * G_+$.

In both cases, the starting path can be incremented. If with the longer path we now reach Y , we stop; otherwise, we iterate the previous procedure. Let us now show that eventually Y will be reached without visiting a vertex twice (and in particular, that Y will be reached in at most $n - d - 1$ steps). Any dual path can be represented as a sequence v_1, \dots, v_m of vertices of C , by associating to each two consecutive facets F_i, F_{i+1} the vertex $F_i -_s F_{i+1}$. Now, along the path we are progressively constructing, $f(v_{i+1}) \geq f(v_i)$. Indeed, since the underlying space of the star of each v_i is convex, we have $f(\text{star } v_i) = f(F_i)$. In particular, after a finite number of steps, we will reach Y . Call Γ the obtained dual path.

If $f(v_i) = f(v_j)$ for some $i < j$, the facets F_i and F_j come from a lifted dual path, and thus differ by induction on the dimension. Thus, the number m of the vertices v_1, \dots, v_m equals the number g of edges of Γ . Now, none of the vertices of Y can appear in the set of the v_i 's, because the vertices of Y maximize f . Therefore, C has at least

$$\#Y + m = (d + 1) + g$$

distinct vertices. Since g is at most $n - d - 1$, this shows that C is Hirsch. Furthermore, the vertices of Y and the vertices v_i are the only vertices visited by Γ in C , and none is encountered twice. Thus, C is also W_v . \square

Recall that a simplicial complex is called *flag* if all its minimal non-faces are edges. In recent literature, complexes all whose faces have pure and strongly connected links are called *superstrongly connected*.

Corollary 4.5. *Let C be a connected, super-strongly connected complex. If C is flag, then it is Hirsch.*

Proof. Let us endow every simplex of C with the metric of a regular spherical simplex with orthogonal dihedral angles. By Gromov's Criterion [43, Theorem 4.2.A], the resulting metric

space is CAT(1), because C is flag. By construction, every simplex of X is nonobtuse in this metric structure, so we can apply Theorem 4.3 with $k = 1$ and conclude that C is Hirsch. \square

Corollary 4.6. *All flag Cohen–Macaulay complexes are Hirsch.*

Corollary 4.7. *All flag homology manifolds are disjoint unions of Hirsch complexes. In particular, the barycentric subdivision of every connected manifold is Hirsch.*

Remark 4.8. The barycentric subdivision of every polytope is vertex decomposable and thus Hirsch, by the results of Provan and Billera [67]. However, the barycentric subdivision of an arbitrary triangulated manifold might not be vertex-decomposable. The reasons are two: There are topological obstructions (all vertex-decomposable manifolds are either balls or spheres) as well as combinatorial obstructions (some spheres with short knots have non-vertex-decomposable barycentric subdivisions, cf. [13]). That said, the barycentric subdivision of any simplicial complex is flag. Now, every connected homology manifold is pure and strongly connected, and it has also pure and strongly connected links. So, by Corollaries 4.5 and 4.7, its barycentric subdivision is Hirsch.

The *Polynomial Hirsch conjecture* (still open!) claims that the diameter of the dual graph of any simplicial d -sphere on n vertices is bounded above by a polynomial in n and d . As noted in a discussion with Eran Nevo, one can modify the proof of Theorem 4.3 to prove a polynomial Hirsch bound for the complexes whose missing faces do not exceed a certain dimension, cf. [63].

Theorem 4.9. *Let a be a positive integer. There is a polynomial $P_a(n, d)$ such that, for every simplicial complex C with*

- *dimension d ,*
- *n vertices,*
- *all face links pure and strongly connected, and*
- *all minimal non-faces of dimension $\leq a$,*

the diameter of the dual graph of C is bounded above by $P_a(n, d)$. Moreover, such polynomial has degree 1 with respect to the variable d and degree a with respect to the variable n .

Proof. The proof is based on the elementary observation that the diameter does not decrease after a subdivision. If $a = 1$, then C is flag, and we are back to the case discussed in Theorem 4.3. Thus, assume $a > 1$. Let N_2 be the set of minimal nonfaces of C of dimension at least 2. For every element σ of N_2 , subdivide every facet τ of σ by barycentric subdivision. This yields a flag subdivision C' of C . There are at most $\binom{n}{a-1}$ such faces τ ; therefore, in passing from C to C' we need to add at most $a!\binom{n}{a-1}$ vertices. By Theorem 4.5, the diameter of C' is bounded above by $a!\binom{n}{a-1} + n - d$. So also the diameter of C is bounded above by $a!\binom{n}{a-1} + n - d$. \square

4.3 A collapsible manifold that is not a ball

A famous result obtained in 1939 by Whitehead [80] is the following:

Theorem 4.10. *Let M be a (compact) manifold. If some PL triangulation of M is collapsible, then M is a ball.*

It remained an open problem whether any compact manifold with a collapsible triangulation has to be a ball. Here we present a negative answer: Some contractible manifolds different than balls may admit collapsible triangulations. Our solution is based on Gromov’s hyperbolization procedure, and it relies heavily on the famous examples by Davis and Januszkiewicz of *non-compact* smooth CAT(0) manifolds different than balls [30, Example 2].

Definition 4.11. Let M be a triangulated manifold. The *PL-singular set* of M is the subcomplex given by the faces of M whose link is not homeomorphic to a sphere or a ball. The *derived neighborhood* of a subcomplex S is the union of all faces in the barycentric subdivision of M that intersect S .

Theorem 4.12. *For each $d \geq 5$, there is a contractible compact d -manifold M such that*

- (i) *M is not homeomorphic to the d -ball;*
- (ii) *M has a smooth structure;*
- (iii) *M admits a decomposition into unit cubes, which supports a CAT(0) metric;*
- (iv) *by subdividing each cube into d -simplices, one obtains a triangulation of M that is collapsible, but not PL.*

Proof. Our proof has two parts:

- (I) We construct M following Davis–Januszkiewicz, cf. [30, Example 2].
- (II) We verify that M has the requested properties.

Part I. *Construction of M á-la-Davis–Januszkiewicz.*

Fix an integer $d \geq 5$. Let C be the manifold obtained by removing the interior of a $(d - 2)$ -simplex from a triangulated homology $(d - 2)$ -sphere, with nontrivial fundamental group. The product complex $C \times [0, 1]$ is a homology ball of dimension $d - 1$. Let us attach to $C \times [0, 1]$ the cone $v * \partial C$, and let B be the resulting complex. The suspension $S = (a * B) \cup (b * B)$ of B is a homology manifold of dimension d . By Edwards’ criterion [34], to prove that S is a manifold we only need to check that the link of every vertex is simply connected. Indeed, the link of an arbitrary vertex w of S is isomorphic to:

- the suspension of $C \times [0, 1]$, if $w = v$;
- the complex B , if w is one of a, b ;
- a $(d - 1)$ -sphere, if w is in $C \times [0, 1]$.

In all cases, the link of w is simply connected. So S is a manifold. Using the Seifert–van Kampen theorem, it is easy to see that S is also simply connected, and has the same homology of the d -sphere. By Smale’s proof of the Poincaré conjecture, S is a d -sphere.

The PL-singular set X of S is the subcomplex induced by the three vertices v, a and b . Let R be a regular neighborhood of X . This R is a contractible submanifold of M . The boundary of R is homeomorphic to the double D of $C \times [0, 1]$. Since $\pi_1(\partial R) = \pi_1(D) = \pi_1(C) \neq (0)$, the neighborhood R cannot be homeomorphic to a ball.

Let us hyperbolize S and then pass to the universal cover. The resulting manifold \tilde{S} has a natural CAT(0) structure. Let \tilde{X} be one of the (multiple) images of the set $X \subset S$ under hyperbolization and universal cover. The manifold M we want is the cubical subcomplex of \tilde{S} given by the faces of \tilde{S} that intersect \tilde{X} .

Part II. *Properties of M .*

- (i) M is homeomorphic to a regular neighborhood of X in S . As we saw earlier, any such neighborhood is a contractible manifold that is not homeomorphic to the d -ball.
- (ii) Since hyperbolizing preserves the smooth structure, and M is homeomorphic to a regular neighborhood of X in S , it suffices to show that this neighborhood has a smooth structure. This can be achieved using Theorem 5.1 (Essay 1) and Theorem 10.1 (Essay 4) in [53]. See also Lashof–Rothenberg [55, Theorem 7.3] or the exposition by Schultz [72].
- (iii) \tilde{X} is a convex set in \tilde{S} , and so is M . Since \tilde{S} is CAT(0), so is every convex subset, and M inherits the canonical structure of cubical CAT(0)-complex.
- (iv) By Theorem 3.16, a cubical CAT(0)-complex is collapsible, and so is M .

□

Corollary 4.13. *Forman’s Discrete Morse Theory is more accurate than smooth Morse Theory in bounding the homology of a manifold.*

Proof. Let M be a topological manifold without boundary that admits some smooth structure (possibly more than one). Then M admits some PL triangulations, and possibly also some non-PL ones. By the work of Gallais [38] (cf. also [10]), any smooth Morse function with c_i critical points of index i induces, on some PL triangulation of M , a discrete Morse function with c_i critical i -simplices. Therefore, any optimal smooth Morse vector is also a discrete Morse vector. The same type of result can be obtained also for manifolds with boundary, cf. [10]. However, an optimal discrete Morse vector can be better than any smooth one, as shown by the vector $(1, 0, \dots, 0)$ for the non-PL collapsible triangulations of Theorem 4.12. \square

Remark 4.14. In [11] the second author has showed that the barycentric subdivision of any collapsible 3-manifold is endo-collapsible. Since every 4-manifold is PL, it follows from [10, 11] that *some* subdivision of any collapsible 4-manifold is also endo-collapsible. However, in dimension five the situation changes radically: our collapsible 5-manifold M from Theorem 4.12 does not admit any endo-collapsible subdivisions, because all endo-collapsible manifolds with non-empty boundary are balls [11].

Remark 4.15. Any Morse function on a smooth d -manifold with c_i critical points of index i induces a handle decomposition of the manifold into c_i d -dimensional i -handles, and the other way around; cf. Milnor [62]. Recently, Jerše and Mramor have partially discretized this, showing that any discrete Morse function on a triangulation C induces a decomposition of C into “descending regions” homeomorphic to balls [49]. As pointed out to the second author by U. Bauer, up to replacing C with its second barycentric subdivision $C' = \text{sd}^2 C$, and up to thickening each region to its regular neighborhood in C' , we can assume that these regions are d -dimensional balls. This suggests a method to obtain a smooth handle decomposition from any given discrete Morse function. However, in the non-PL case, the method might result in a handle decomposition of something *homotopy equivalent to*, but not necessarily *homeomorphic to*, the starting triangulation. For example, if C is the non-PL collapsible triangulations of Theorem 4.12, the “descending region” of C consists of a single vertex. By thickening it to a regular neighborhood, we obtain a d -ball B' inside $C' = \text{sd}^2 C$. However, even if C' collapses onto B' , the two manifolds C' and B' are not homeomorphic. We are grateful to Ulrich Bauer and Neža Mramor for helpful conversations on this topic.

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References

- [1] K. ADIPRASITO AND I. PAK, Acute triangulations of \mathbb{R}^4 . In preparation (2011).
- [2] S. ALEXANDER AND R. BISHOP, The Hadamard–Cartan theorem in locally convex metric spaces. *Enseign. Math.* (2) 36 (1990), 306–320.
- [3] N. ALON, The number of polytopes, configurations and real matroids. *Mathematika*, 33 (1986), 62–71.
- [4] J. AMBJØRN, B. DURHUUS AND T. JONSSON, Quantum geometry. Cambridge Univ. Press (1997).
- [5] F. ARDILA, M. OWEN AND S. SULLIVANT, Geodesics in CAT(0) cubical complexes. *Adv. Appl. Math.*, to appear. Preprint at [arXiv:1101.2428](https://arxiv.org/abs/1101.2428).

- [6] W. BALLMANN, Singular spaces of non-positive curvature, In *Sur les groupes hyperboliques d'après Mikhael Gromov* (E. Ghys, P. de la Harpe, eds.), Birkhäuser, Boston, 1990, Chap. 10, 189–202.
- [7] H. J. BANDELT AND V. CHEPOI, Metric graph theory and geometry: a survey. *Surveys on discrete and computational geometry: Twenty years later*. *Contemp. Math.* 453 (2008), 49–86.
- [8] J. A. BARMAN AND G. E. MINIAN, Strong homotopy types, nerves and collapses. *Discrete Comput. Geom.*, to appear. Preprint at [arXiv:0907.2954](#).
- [9] C. BARTOCCI, U. BRUZZO, M. CARFORA AND A. MARZUOLI, Entropy of random coverings and 4D quantum gravity. *J. Geom. Physics* 18 (1996), 247–294.
- [10] B. BENEDETTI, Discrete Morse theory is as perfect as Morse theory. Preprint at [arXiv:1010.0548](#).
- [11] B. BENEDETTI, Discrete Morse theory for manifolds with boundary. *Trans. Amer. Math. Soc.*, to appear. Preprint at [arXiv:1007.3175](#).
- [12] B. BENEDETTI AND F. H. LUTZ, Knots in collapsible and non-collapsible balls. In preparation (2011).
- [13] B. BENEDETTI AND G. M. ZIEGLER, On locally constructible spheres and balls. *Acta Mathematica* 206 (2011), 205–243.
- [14] L. BILLERA, S. HOLMES AND K. VOGTMANN, Geometry of the space of phylogenetic trees. *Adv. Appl. Math.* (2001), 733–767.
- [15] R. H. BING, Some aspects of the topology of 3-manifolds related to the Poincaré conjecture. In *Lectures on Modern Mathematics* (T. Saaty ed.), vol. II, Wiley, 1964, 93–128.
- [16] L. BOMBELLI, A. CORICHI AND O. WINKLER, Semiclassical quantum gravity: statistics of combinatorial Riemannian geometries. *Ann. Phys.* 14 (2005), 499–519.
- [17] H. BRUGGESSER AND P. MANI, Shellable decompositions of cells and spheres. *Math. Scand.* 29 (1971), 197–205.
- [18] D. BURAGO, Y. BURAGO AND S. IVANOV, A course in metric geometry. *Graduate Studies in Mathematics* 33, AMS, Providence, 2001.
- [19] Y. BURAGO, M. GROMOV AND G. PERELMAN, A. D. Alexandrov spaces with curvature bounded below (part 1), *Russian Math. Surveys* (2) 47 (1992), 1–58.
- [20] J. CHEEGER, Comparison theorems in Riemannian geometry. *North Holland Mathematical Library*, Elsevier, 1975.
- [21] J. CHEEGER, Critical points of distance functions and applications to geometry. In *Geometric Topology: Recent Developments. Lecture Notes in Mathematics*, Springer, 1991.
- [22] J. CHEEGER, Finiteness theorems for Riemannian manifolds. *Amer. J. Math.* 92 (1970), 61–74.
- [23] V. CHEPOI, Graphs of some CAT(0) complexes. *Adv. Appl. Math.* 24 (2000), 125–179.
- [24] V. CHEPOI AND D. OSAJDA, Dismantlability of weakly systolic complexes and applications. Preprint at [arxiv:0910.5444](#).
- [25] D. R. J. CHILLINGWORTH, Collapsing three-dimensional convex polyhedra. *Proc. Camb. Phil. Soc.* 63 (1967), 353–357. Erratum in 88 (1980), 307–310.
- [26] M. M. COHEN, Dimension estimates in collapsing $X \times \mathbb{I}^q$. *Topology* 14 (1975), 253–256.
- [27] Y. COLIN DE VERDIÈRE, Comment rendre géodésique une triangulation d’une surface? *Enseign. Math.* 37 (1991), 201–212.
- [28] R. CONNELLY AND D. W. HENDERSON, A convex 3-complex not simplicially isomorphic to a strictly convex complex. *Math. Proc. Cambridge Phil. Soc.* 88 (1980), 299–306.
- [29] K. CROWLEY, Simplicial collapsibility, discrete Morse theory, and the geometry of nonpositively curved simplicial complexes. *Geometriae Dedicata* 133 (2008), 35–50.
- [30] M. W. DAVIS AND T. JANUSZKIEWICZ, Hyperbolization of polyhedra. *J. Diff. Geom.* 34 (1991), 347–388.
- [31] M. W. DAVIS AND G. MOUSSONG, Notes on nonpositively curved polyhedra, in *Low dimensional topology* (K. Boroczky, W. Neumann, A. Stipsicz, eds.), Bolyai, 1999.
- [32] B. DURHUUS AND T. JONSSON, Remarks on the entropy of 3-manifolds. *Nucl. Phys. B* 445 (1995) 182–192.
- [33] R. D. EDWARDS, The double suspension of a certain homology 3-sphere is S^5 . *Notices AMS* 22 (1975), 334.
- [34] R. D. EDWARDS, The topology of manifolds and cell-like maps. *Proc. ICM 1978 Helsinki*, 111–127.
- [35] D. EPPSTEIN, J. M. SULLIVAN AND A. ÜNGÖR, Tiling space and slabs with acute tetrahedra. *Comput. Geom.* 27 (2004), 237–255.
- [36] R. FORMAN, A user’s guide to discrete Morse theory. *Sem. Lothar. Comb.* 48: Art B48c (2002).
- [37] R. FORMAN, Morse theory for cell complexes. *Adv. in Math.* 134 (1998), 90–145.

- [38] E. GALLAIS, Combinatorial realization of the Thom–Smale complex via discrete Morse theory. *Ann. Sc. Norm. Super. Pisa, Cl. Scienze* (5) 9, 229–252 (2010).
- [39] S. M. GERSTEN, Dehn functions and l_1 -norms of finite presentations. In *Algorithms and classification in combinatorial group theory*. MSRI Publications, Springer, 1991.
- [40] S. M. GERSTEN, Isoperimetric and isodiametric functions of finite presentations. In *Geometric Group Theory*. LMS Lecture Note Series 181, Cambridge Univ. Press, 1993.
- [41] J. E. GOODMAN AND R. POLLACK, There are asymptotically far fewer polytopes than we thought. *Bull. Amer. Math. Soc.*, 14 (1986), 127–129.
- [42] R. E. GOODRICK, Non-simplicially collapsible triangulations of I^n . *Proc. Camb. Phil. Soc.* 64 (1968), 31–36.
- [43] M. GROMOV, Hyperbolic groups, in *Essays in Group Theory*, MSRI Publications, Springer, 1987.
- [44] M. GROMOV, Spaces and Questions. *Geom. Funct. Anal.*, special volume, Part I (2000), 118–161.
- [45] K. GROVE, P. PETERSEN AND J. Y. WU, Geometric finiteness theorems via controlled topology, *Inventiones Math.* 99 (1990), 205–213.
- [46] M. HACHIMORI AND G. M. ZIEGLER, Decompositions of simplicial balls and spheres with knots consisting of few edges. *Math. Z.*, 235 (2000), 159–171.
- [47] F. HAGLUND, Complexes simpliciaux hyperboliques de grande dimension, *Prép. Orsay* 71 (2003). Available at www.math.u-psud.fr/~biblio/ppo/2003/fic/ppo_2003_71.pdf.
- [48] T. JANUSZKIEWICZ AND J. ŚWIĄTKOWSKI, Simplicial nonpositive curvature. *Pub. Math. de l’IHES* 104 (2006), 1–85.
- [49] G. JERŠE AND N. MRAMOR KOSTA, Ascending and descending regions of a discrete Morse function. *Comput. Geom.* 42 (2009), 639–651.
- [50] J. KAHN, M. SAKS AND D. STURTEVANT, A topological approach to evasiveness. *Combinatorica* 4 (1984), 297–306.
- [51] G. KALAI, Many triangulated spheres. *Discrete Comput. Geom.*, 3 (1988), pp. 1–14.
- [52] R. C. KIRBY, Problems in low-dimensional topology (1995). Available online at <http://math.berkeley.edu/~kirby/>.
- [53] R. C. KIRBY AND L. C. SIEBENMANN, Foundational essays on topological manifolds, smoothings and triangulations. *Ann. of Math. Studies* 88, Princeton University Press, 1977.
- [54] E. KOPCZYNSKI, I. PAK AND P. PRZYTICKI, Acute triangulations of polyhedra and R^n . *Combinatorica*, to appear. Preprint at [arXiv:0909.3706](https://arxiv.org/abs/0909.3706).
- [55] R. LASHOF AND M. ROTHENBERG, Microbundles and smoothing. *Topology* 3 (1965), 357–388.
- [56] W. B. R. LICKORISH, Unshellable triangulations of spheres. *Europ. J. Combin.* 12 (1991), 527–530.
- [57] W. B. R. LICKORISH AND J. M. MARTIN, Triangulations of the 3-ball with knotted spanning 1-simplexes and collapsible r -th derived subdivisions. *Trans. Amer. Math. Soc.* 170 (1972), 451–458.
- [58] F. H. LUTZ, Examples of \mathbb{Z} -acyclic and contractible vertex-homogeneous simplicial complexes. *Discrete Comput. Geom.* 27 (2002), 137–154.
- [59] P. MANI AND D. W. WALKUP, A 3-sphere counterexample to the W_v -path conjecture. *Math. Oper. Res.* 5 (1980), 595–598.
- [60] J. MATOÚŠEK, *Lectures in discrete geometry*. Springer, Graduate Texts in Math. 212, 2002.
- [61] B. MAZUR, A note on some contractible 4-manifolds. *Annals of Math.* 73 (1961), 221–228.
- [62] J. MILNOR, *Lectures on the h -cobordism theorem*. Princeton University Press, 1965.
- [63] E. NEVO, Remarks on missing faces and lower bounds on face numbers. *Electr. J. Comb.* 16 (2009).
- [64] M. NEWMAN, Boundaries of ULC sets in Euclidean n -space. *Proc. Nat. Acad. Sci.* 34 (1948), 193–196.
- [65] Y. OTSU AND T. SHIOYA, The Riemannian structure of Alexandrov spaces, *J. Differential Geom.*, 39 (1994), 629–658.
- [66] J. PFEIFLE AND G. M. ZIEGLER, Many triangulated 3-spheres. *Math. Annalen* 330 (2004), 829–837.
- [67] J. S. PROVAN AND L. J. BILLERA, Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Operations Research* 5 (1980), 576–594.
- [68] T. REGGE, General relativity without coordinates. *Il Nuovo Cimento* 19 (1961), 558–571.
- [69] C. P. ROURKE AND B. J. SANDERSON, *Introduction to piecewise-linear topology*. Springer, 1972.
- [70] M. E. RUDIN, An unshellable triangulation of a tetrahedron. *Bull. Amer. Math. Soc.* 64 (1958), 90–91.
- [71] F. SANTOS, A counterexample to the Hirsch conjecture. *Ann. of Math.*, to appear. Preprint at [arXiv:1006.2814](https://arxiv.org/abs/1006.2814).
- [72] R. SCHULTZ, Smoothable submanifolds of a smooth manifold. Preprint at <http://math.ucr.edu/~res/miscpapers/smoothablesubmflds.pdf>.

- [73] V. V. SHARKO, Functions on manifolds: Algebraic and topological aspects. Transcr. Math. Monographs 131, AMS, 1993.
- [74] W. T. TUTTE, A census of planar triangulations. Canadian J. Math, 14 (1962), 21–38
- [75] E. VANDERZEE, A. N. HIRANI, V. ZHARNITSKY AND D. GUOY, A dihedral acute triangulation of the cube. Comput. Geom., to appear. Preprint at [arXiv:0905.3715](https://arxiv.org/abs/0905.3715).
- [76] D. W. WALKUP The Hirsch conjecture fails for triangulated 27-spheres. Math. of Oper. Res. 3 (1978), 224–230.
- [77] D. WEINGARTEN, Euclidean quantum gravity on a lattice. Nuclear Phys. B, 210 (1982), 229–245.
- [78] V. WELKER, Constructions preserving evasiveness and collapsibility. Discr. Math. 207 (1999), 243–255.
- [79] J. H. C. WHITEHEAD, A certain open manifold whose group is unity. Quart. J. Math. Oxford 6 (1935), 268–279.
- [80] J. H. C. WHITEHEAD, Simplicial spaces, nuclei and m -groups. Proc. Lond. Math. Soc., 45 (1939), 243–327.
- [81] E. C. ZEEMAN, On the dunce’s hat. Topology, 2 (1964), 341–358.
- [82] G. M. ZIEGLER, Lectures on polytopes. Springer, Graduate Texts in Math. 152, 1998.